5. "Some elementary aspects of Heaviside's operational methods" by Professor H. J. ETTLINGER, University of Texas.

6. "N-ic surfaces in S_3 generated by triples of N-ic involutions on a line" by Mr. F. AYRES, Texas Agricultural and Mechanical College.

7. "The golden section" by Dean J. J. QUINN, St. Edward's University.

8. "A prescribed course for freshmen and their mathematical habits," by Professor H. E. BRAY, Rice Institute.

9. "Preparation for College mathematics" by Professor Chas. E. PURYEAR, Dean of the College, Texas Agricultural and Mechanical College (by invitation).

Abstracts of these papers follow:

1. Professor Ford exhibited a line-conic camera, in which the lens is replaced by two slits. The image of a point moving along a line will be a conic section, since the photographic plate intersects the locus of a line which touches three skew lines. Several interesting photographs were exhibited taken by means of this camera.

2. The notions of accessibility and *regular accessibility* of the boundary points of a domain, as well as certain accessibility propositions which hold only in the plane, were discussed by Professor Whyburn with a view of possible extensions to 3-space. A certain "tubular accessibility" in 3-space was offered which seems to carry a number of the well known theorems in the plane about simple continuous arcs and arc-wise accessibility over into higher dimensions. The relation between the regular accessibility of the boundary points of a domain and the connectedness im kleinen of the domain plus certain subsets of its boundary was also discussed. It was shown that although there exists a plane domain having an isolated non-accessible boundary point, each non-*regularly* accessible boundary point of a plane domain must belong to a continuum of such boundary points.

3. The paper by Mr. Blau appeared in full in the June-July 1928 issue of this MONTHLY.

4. Let T be a three dimensional open simply connected region on the bounday of which the Green's function with pole at an interior point O vanishes. Let T be mapped upon the interior T' of the unit sphere, a level surface corresponding to a surface of a sphere, and an orthogonal trajectory to a radius of T'. Let A be an accessible point on the boundary of T, and OA the curve leading to A. Then if OA_t be any trajectory leading to A such that the curve $OA+OA_t$ can be shrunk down to A without leaving T, the image of OA_t in T' is a radius which terminates at some one of the points of a set E of zero measure.

5. In his operational calculus for solving linear differential equations, Heaviside makes use of a simple discontinuous function, called the *unit function* and defined as follows: $f(t)=0$, when $t\leqq 0$, $f(t)=1$, when $t>0$. This is an elementary type of horizontal or step function making use only of the values 0

and 1. Professor Ettlinger discussed some of the properties of the operator p^{-1} operating on the unit function which is equivalent to integration between the limits minus infinity and t. Positive integral powers of this operator generate the integral power function and linear combinations generate polynomials in t. Conditions were derived under which a power series in this operator will generate an analytic function of t over the entire t-axis.

6. In this paper, Mr. Ayres has shown that certain nondegenerate n-ic surfaces can be generated by means of triples of n-ic involutions, or linear series of points; namely the first polar surface of any one with respect to the n-ic degenerates into $n-1$ planes through the line determined by the other two. These points and the lines determined by them lie on the first, $\cdots$, $(n-3)$rd Hessian and corresponding Steinerian surfaces. For the case $n=2$, it was shown that every quadric surface can be generated in the above manner.

7. Dean Quinn presented a new construction for locating the golden section point of a given segment. This paper will appear in full in a fall number of the Texas Mathematics Teachers' Bulletin, published by the University of Texas.

8. Professor Bray outlined a method of handling freshman students in mathematics at Rice Institute which consists of lectures and "supervised" working of problems. Professor Bray gave a summary of the characteristic ailments of freshman mathematical students. This paper will appear in full in a fall number of the Texas Mathematics Teachers' Bulletin.

9. Dean Puryear presents a summary of what the high school graduate should know of mathematics upon entrance to college. He also made a plea for a course in algebra in the last year of the high school course. The paper appears in full in the May 1928 number of the Texas Mathematics Teachers' Bulletin.

H. J. ETTLINGER, *Secretary*

THE TWELFTH ANNUAL MEETING OF THE KENTUCKY SECTION

The twelfth annual meeting of the Kentucky Section of the Mathematical Association of America was held April 21, 1928 at the University of Louisville, Louisville, Ky. Professor Guy Stevenson presided at both sessions. At noon a delightful dinner was served in the Home Economics Building.

At the business meeting the minutes of the previous meeting were read and approved. Professor C. G. LATIMER, University of Kentucky, was elected chairman for the year 1928–1929. Professor A. R. FEHN, Centre College, was elected secretary for the five years, 1928–1933. A vote of thanks and appreciation was given Professor R. D. Carmichael, University of Illinois, for his lecture

and his interest in the Kentucky Section. Resolutions, drawn up by P. P. Boyd, of the University of Kentucky, expressing regret that the Section is to lose Professor C. H. Richardson, Georgetown College, were passed; Professor Richardson will be at Bucknell, Penn., next year. The Section expressed its appreciation to the University of Louisville for the hospitable reception and the delightful dinner given the attending members.

There were twenty eight present including the following members: Dean P. P. Boyd, J. L. Clayton, J. M. Davis, D. G. Dearman, H. H. Downing, A. R. Fehn, R. E. Hill, C. G. Latimer, E. L. Rees, C. H. Richardson, G. E. Stevenson, B. B. Vance.

Papers were presented as follows:

1. "The Steinerian transformation" by Dean P. P. BOYD, University of Kentucky.

2. "Resolutions of singularities" by Professor M. B. TOLAR, Georgetown College, (by invitation).

3. "On the representation of integers by indefinite forms" by C. G. LATIMER, University of Kentucky.

4. "Curves of accumulation" by Professor H. H. DOWNING, University of Kentucky.

5. "Relativity" by Professor R. D. CARMICHAEL, University of Illinois.

6. "Correlation between mental and mathematical tests and first semester grades" by Miss LYDIA FREMD, University of Kentucky (by invitation).

7. "Classification of freshmen" by Professor R. E. HILL, University of Louisville.

8. Business Meeting.

Abstracts of these papers follow:

1. Dean Boyd presented for examination a copy of the Elements of Euclid, done in vulgar tongue under the direction of the most famous Mathematician Federigo Commandinus of Urbino,—carefully revised to date—printed by Flaminio Concordia,—MDCXIX—under the direction of George Antonio Tugegnieri of Fossombrone." The volume was given to the speaker by a friend who found it in Spain a year or two ago. Selections were quoted from the lengthy prologue, setting forth the nature of mathematics, its uses and beauties, and defending the name of Euclid from his detractors. Sample demonstrations were also given.

2. This paper outlined briefly the analytical development of the theory of quadratic inversion and its application to higher singularities. The geometrical aspect was presented in an analysis of the singularities on the curve $x^4y^3=z^7$.

3. In this paper Professor Latimer considered the forms where a is a positive integer less than 100 with no square factor greater than 1 and in the form $8n+5$

or $8n+7$. By a method used in a former paper (Annals of Mathematics, vol. 28) it was shown that each of these forms represents every integer.

4. In this paper attention was directed to the direct method of solving the problem of the calculus of variations. The principal thought in the paper was centered in the theorems of Ascoli, Argela, and Hilbert in which the existence of functions and curves of accumulation are proved. The treatment was that of Tonelli in *Fondimenti di calcolo delle variationi.*

5. Professor Carmichael outlined briefly the historical development of the theory of relativity, sketched its foundations in non-technical language, discussed rapidly its general bearing on the development of science, and analyzed its influence on philosophical, scientific, and ethical thought.

6. The Problem in question form:

(a) What is the relation of the mathematical and mental-test scores made by the University of Kentucky freshmen of 1927 to the first semester grades in College Algebra made by the same students?

(b) What weight, discovered by means of the regression equation, should be attached to each score to obtain the optimum correlation?

(c) What is the test, or the elements of the test, which furnish the best prognosis of the standing of the students in mathematics?

Data: The study includes 140 cases, a wholly unselected group, as it includes every student who had a grade recorded with the registrar for the first semester of college algebra and whose test records were complete. Tests were given under the supervision of the Psychology Department. They were Woody-McCall, Mixed fundamentals, Otis mathematics tests, Reasoning, Iowa training tests, Iowa high school content, College classification and Otis mental tests.

Results: Median quartile scores for all tests with grade groups.

	W-Mc	Otis	Iowa Tr.	Iowa H. S.	Mental	Combined
A	1.32	1.44	2.30	2.15	1.95	1.65
B	1.02	.95	1.48	1.33	1.32	1.15
C	.69	.91	1.20	.65	.93	.80
D	.24	.51	.27	.34	−.17	.15
E	−.15	−.50	.15	−.55	−.85	−.30

The median quartile scores descend by a gradual consistent decline as the grades pass from a higher to a lower grade, which points to an unequivocal relationship between test scores and grades. No median of any test is greater than, or equal to, the median of a higher grade.

A. R. FEHN, *Secretary*

THE FIFTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held in Jackson, Miss., March 30–31, 1928, Professor S. T. Sanders of Louisiana State University presiding.

The following is the list of the delegates at the meeting: H. E. Buchanan, J. B. Cole, Mrs. A. P. Daspit, W. L. Duren, Jr., J. E. Hardin, J. R. Hitt, Mabel Hutchins, George Litchfield, I. Maizlish, B. E. Mitchell, C. N. Moore, I. C. Nichols, R. L. O'Quinn, W. C. Roaten, W. J. Rothfuss, S. T. Sanders, C. Shroeder, C. D. Smith, H. L. Smith, P. K. Smith, N. Smylie, W. B. Stokes, J. B. Temple, J. F. Thompson, N. Touchstone, B. A. Tucker, B. M. Walker, J. F. Welch, W. P. Webber, R. S. Wynn. In addition to the above thirty delegates, there were a few who failed to register.

The meeting of the Section was held in conjunction with the meeting of the Louisiana-Mississippi Section of the National Council of Teachers of Mathematics. On the afternoon of the thirtieth, at a joint meeting of the two sections, several papers were read bearing upon high school mathematics. The college teachers had an opportunity to hear first hand statements on problems in high school mathematics teaching.

At Mississippi College, on the evening of the thirtieth, a dinner was tendered to both sections. Dr. C. N. MOORE, of the University of Cincinnati, the speaker of the evening, spoke upon "Mathematics and Civilization." The speaker cited many interesting cases from the history of science where the scientist has had to turn to the mathematician to give the final word in the solution of his problem. President B. M. Walker, of the A. and M. College of Mississippi, President J. W. Provine, of Mississippi College, and Chairman Sanders also spoke.

At a joint meeting of the sections, on the morning of the thirty-first at Belhaven College, five papers from the college group were read. Following this session a business meeting was held by the section of the Association. Prof. B. E. MITCHELL, of Millsaps College, was elected chairman; Professor S. T. SANDERS, of Louisiana State University, Editor of the "Mathematics News Letter"; Professor J. R. HITT, of Mississippi College, and Professor W. B. STOKES, Vice-chairmen; and Professor J. P. COLE, of Louisiana Polytechnic Institute, Secretary-Treasurer.

Upon adjournment of the business meeting Millsaps College tendered both sections a barbecue.

The titles of the papers read are as follows:

1. "A quadratic one-to-one transformation between two planes," by President B. M. WALKER, A. and M. College of Mississippi.

2. "The construction of magic squares," by Professor J. R. HITT, Mississippi College.

3. "A problem in geometry," by Professor J. P. COLE, Louisiana Polytechnic Institute.

4. "Some remarks on periodic functions," by Professor PAUL WEBBER, Louisiana State University.

5. "A table of chords by Ptolemy," by Professor H. E. BUCHANAN, Tulane University.

Abstracts of these papers follow:

1. The transformation system of equations was first deduced to be $x'\colon y'\colon z'=x^2+yz\colon xz\colon xy$, and following immediately the system of inverse equations was deduced. The treatment of the problem included ordinary points, related points, fundamental points, transition curve or "Uebergangs Curve," and the double curve or "Doppia Curva." Related points were shown to be connected by a quadratic Cremona one-to-one correspondence. It was further shown that to a fundamental point together with the totality of the paths of approach to it in one plane corresponds a fundamental line in the other plane. The one-to-two correspondence was then established by means of projection using a quadric surface and two centers of projection arbitrarily chosen so that one should lie on the surface and the other be external to the surface. The one-to-two correspondence was further established by means of poles and polars. The problem of a one-to-two correspondence was reduced to the problem: given a pole of a cubic to find the polar line; and the line coordinates defined in terms of the coordinates of the given pole are the equations of transformation. The inverse problem being: given a polar line to find its two poles.

2. The author showed (*a*) a general method of constructing magic squares whose square roots are odd; (*b*) a general method of constructing squares whose square roots are multiples of 4; (*c*) special methods of constructing squares whose square roots are multiples of 3 or multiples of 4.

3. The author divided his discussion into three parts. The problem proper was to construct a square so that each side would contain a given point. In the first part four points were taken and connected by straight lines. These lines were then bisected and these points were then connected, forming a parallelogram. On the longer diagonal (assuming one longer than the other) a circle was constructed with it as a diameter. The arc on one side of the diameter was then bisected. This point and one of the other points of the parallelogram were connected and the line extended to the other arc of the circle. This other intersection of the line with the circle was then joined to the end of the diameter of the circle. Through two of the original points, lines were drawn parallel to this last line, and through the other two points lines were drawn perpendicular to the line. The figure so constructed was the required square. In the second part it was shown that if the parallelogram was a rectangle it would have to be a square in order that a square might be drawn through the original

points. In the third part it was shown that if the rectangle was a square an infinite number of squares could be drawn through the four original points.

4. Mr. Webber in his remarks on periodic functions presented his generalized formulae defining functions having n periods and referring to space of n dimensions and called attention to the analogy of form with the fundamental formulae of the Weierstrassian theory and further showed that in the case of $n=2$ his formula reduces identically to Weierstrass's; and when $n=1$, to an analogous form for the cotangent.

5. Professor Buchanan's paper was an attempt to explain the work of Ptolemy on chords. A statement of his results was given in modern notation which showed that he was in possession of practically all of computational trigonometry.

P. K. SMITH, *Secretary*

THE FOURTH ANNUAL MEETING OF THE MICHIGAN SECTION

The fourth annual meeting of the Michigan Section of the Mathematical Association of America was held at Ann Arbor, April 29, 1927.

The following officers were elected for the year: A. L. NELSON, College of the City of Detroit, Chairman; W. W.. DENTON, University of Michigan, Secretary-Treasurer; T. O. WALTON, Kalamazoo College, Member of the Executive Committee.

The following program was presented:

1. "Areas on the sphere" by Professor G. Y. RAINICH.
2. "Marginal notes" by Professor T. H. HILDEBRANDT.
3. "An application of Thiele's semi-invariants to the sampling problem" by Professor C. C. CRAIG.
4. "Focal point construction for a certain type of problem in the calculus of variations" by Professor V. G. GROVE.
5. "A convergence proof in the theory of linear differential equations" by Professor J. A. NYSWANDER.

W. W. DENTON, *Secretary*

THE NINTH ANNUAL MEETING OF THE ILLINOIS SECTION

The ninth annual meeting of the Illinois Section of the Mathematical Association of America was held at Eastern Illinois State Teachers College, Charleston, on May 4–5, 1928. The chairman, Professor C. N. Mills, presided. There was an attendance of 51 at the meetings, including the following 26 members of the Association: Beulah Armstrong, Edith I. Atkin, H. W. Bailey,

O. E. Brown, R. D. Carmichael, C. E. Comstock, D. R. Curtiss, Elinor B. Flagg, A. E. Gault, Frances Harshbarger, Martha Hildebrandt, E. C. Kiefer, J. M. Kinney, Harry Levy, E. B. Miller, C. N. Mills, Mary W. Newson, J. A. Nyberg, H. P. Rogers, H. A. Simmons, H. E. Slaught, C. H. Smiley, E. H. Taylor, Mildred E. Taylor, C. A. Van Velzer, and Alice Winbigler.

In the absence of the Secretary, Professor Newson was appointed secretary *pro tem.* The following officers were elected for the ensuing year: A. E. GAULT, Bradley Polytechnic Institute, Chairman; C. I. PALMER, Armour Institute of Technology, Vice-Chairman; C. N. MILLS, Illinois State Normal University, Secretary-Treasurer.

The next meeting will be held on May 3–4, 1929, at Carthage College, Carthage, Illinois. A motion was passed that the incoming secretary-treasurer pay Professor Rietz, from the section treasury, for his travelling expenses incurred for the meeting in May, 1927.

Professor Mills proposed a plan for making it easier for members to pay their sectional dues. A vote of thanks to the Eastern State Teachers College was passed.

The following papers were read:

1. "Derivatives whose orders are not positive integers," by Professor C. E. VAN VELZER, Carthage College.

2. "Some mathematical aspects of public utilities management," by Dr. A. R. SCHWEITZER, Chicago.

3. "The ideals of the teacher of mathematics," by Professor MARY NEWSON, Eureka College.

4. "The typewriting of mathematical lectures," by Mr. ORLEY E. BROWN, Northwestern University.

5. "Recent developments in differential geometry," by Professor HARRY LEVY, University of Illinois.

6. "Stepping stones," an address open to the public, by Professor R. D. CARMICHAEL, University of Illinois.

7. "Expository paper on certain boundary value problems," by Dr. F. R. BAMFORTH, University of Chicago (by invitation).

8. "Plane curves defined by certain projectivities," an expository presentation, by Miss MARTHA HILDEBRANDT, Proviso High School, Maywood, Illinois.

9. "An expository paper on analysis situs," by Dr. H. R. BRAHANA, University of Illinois (by invitation).

10. "Symposium on the teaching of algebra": (*a*) "Continuation of previous report," by Professor C. E. COMSTOCK, Bradley Polytechnic Institute; (*b*) "Some special phases of the subject," by Mr. J. A. NYBERG, Hyde Park High School, Chicago, Illinois.

Abstracts of these papers follow:

1. In this paper an effort was made to find definitions which would give meanings to derivatives whose orders were negative integers or positive or negative fractions, these definitions to conform to the index and distributive laws and at the same time to be consistent throughout the whole range of functions. Such definitions were found which, in the case of transcendental functions were usually expressed in a series whose terms contained gamma functions. In some cases, however, the fractional derivatives of transcendental functions were expressible in a finite form. Charts were shown of a number of fractional derivatives and some of the properties of these curves were pointed out.

2. Dr. Schweitzer's paper had to do with the isolation of the guiding principles of the organization and operation of the light and power corporations, and with the investigation of their mathematicological significance from a theoretical standpoint. He shows that these guiding principles are the same as those of mathematical sciences, as established in his dissertation. This identity leads naturally to the concept of continuities underlying the discrete data presented by the statistical reports of the light and power corporations. The study of such continuities at the present time centers about the modern mathematical investigations into biological mass formation and growth, with application to monopoly and holding companies.

3. A preliminary study seeking what objectives are common to all stages of the study of mathematics. Certain illustrations were given tending to show that the great features of such a system of objectives are the same, whether the child leaves the school early or continues the study of these branches through college or perhaps beyond. Mrs. Newson also gave other illustrations from her experience in teaching, showing the need of more care in grafting the new subjects on the old, as of algebra on arithmetic, and of continual vigilance to avoid the danger of a superficial study of processes which leaves the student unable to apply what he has learned.

4. Mr. Brown has developed two appliances for the Hammond typewriter which facilitate its use in typewriting mathematics. One of these is a new type set or shuttle designed to be supplementary to 135 C, which is the principal mathematical shuttle now available. Shuttles now carried for sale were designed as complete units, but either overlap almost completely or else have different styles of type. Any one shuttle is limited to 120 characters, so that one generally adequate shuttle is impossible. The new shuttle typewriter together with the old one carries 240 characters, with no duplication excepting punctuation. The other appliance is a shift key placed in the keyboard which interchanges the two sets of type in the machine.

5. Levi-Civita, when he introduced the concept of parallelism in a Riemann space, made a contribution of fundamental importance to geometry, and the

development in the succeeding ten years of Non-Riemannian geometries has been a direct consequence of his investigations.

In an n-dimensional space of coordinates $x^1, x^2, \cdots x^n$, an arbitrary analytic transformation of those coordinates

$$x^i = f^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n) \qquad i = 1,2, \cdots, n,$$

induces a linear transformation on the differentials,

$$dx^i = \sum_{j=1}^{n} \frac{\partial f^i}{\partial \bar{x}^j} d\bar{x}^j,$$

Since a set of differentials $(dx^1, dx^2, \cdots, dx^n)$ defines a direction, the relations among directions at a point are those of projective geometry.

But there is *a priori* no association of directions at different points. One can by postulating a Riemann metric,

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j,$$

define, as did Levi-Civita, parallelism of vectors. There is however no logical justification for subordinating the concept of parallelism to that of distance, and the work of Weyl, Eisenhart and Veblen, Schouten, Cartan, and others has been to take as the starting point a definition parallelism of vectors by a means of linear connection

$$d\xi^i = \sum_{j,k=1}^{n} \Gamma^i_{jk} \xi^i dx^k \qquad i = 1,2, \cdots, n,$$

where Γ^i_{jk} are not the Christoffel symbols of the Riemannian metric but arbitrary functions of the coordinates.

A rather extensive bibliography on the subject may be found in the two recent works of L. P. Eisenhart, *Riemannian Geometry*, (Princeton University Press), and *Non-Riemannian Geometry*, (American Mathematical Society).

6. Professor Carmichael discussed various mathematical activities, such as clubs, national and sectional meetings of the Mathematical Association of America, meetings of the American Mathematical Society, and the publication and the purchase of books, as successive stepping stones to the higher mathematical achievements and to some of the more profound values of human life.

7. This paper is an expository account of some of the important topics in the theory of boundary value problems. The discussion is confined to problems in which there is a system of two ordinary linear homogeneous differential equations of the second order, and a set of two algebraic equations which are linear and homogeneous in the values of the two dependent variables at the end points of the interval $a \leqq x \leqq b$. The coefficients of the differential equations

have continuous first derivatives with respect to x on the interval (ab), and contain a parameter λ linearly; the coefficients of the boundary equations are constants. In the study of a boundary value problem it is necessary to find out whether there exist values of λ for which there are non-trivial solutions of the differential equations satisfying the boundary conditions. Such parameter values, if they exist, are called the characteristic numbers of the problem, and the non-trivial solutions, which correspond to them, the characteristic solutions. Other subjects of interest in the study of boundary value problems of the type mentioned above are the so-called Green's matrix, self-adjoint boundary value problems, the reduction of problems to normal forms, the oscillations of the characteristic solutions, and the expansion of a set of two arbitrary functions simultaneously in terms of these solutions.

8. This paper discusses the locus of intersection of the corresponding lines or the envelope of the lines joining the corresponding points when a projectivity has been established between the elements of two one-dimensional forms of the first, second, or higher order in a plane.

9. The paper by H. R. Brahana explained the position of analysis situs in relation to Euclidean and projective geometries, contained a brief mention of some of the famous problems, and pointed out some of the difficulties encountered in passing from two dimensional analysis situs to that of a higher number of dimensions.

10 (*a*). The history of the entrance requirements in mathematics in American colleges was sketched. Attention was called to the change in entrance requirements as pending before the colleges of the North Central Association, reducing the total number of units from 15 to 12 which comprise the work of the senior high schools. The question at issue was the effect of this new legislation upon the status of algebra as a college entrance requirement.

10 (*b*). This paper was published in the *Mathematics Teacher*, Dec., 1927, and is a discussion of a paper with the same title by Professor W. D. Reeve of Columbia University, published in the *Mathematics Teacher*, March, 1927. While the latter paper calls attention to all the desirable objectives in a third semester's work in algebra, Mr. Nyberg discusses how many of these objectives can be attained in a semester's work of 89 days. Attention was also called to the fact that colleges allow their students to register for courses in trigonometry and in college algebra even when the student has had only one year of algebra in high school. On the other hand no high school allows a pupil to take these courses unless the pupil has completed one and one half years of algebra.

MARY W. NEWSON, *Secretary pro tem.*

THE HUMAN SIGNIFICANCE OF MATHEMATICS[1]

By DUNHAM JACKSON, University of Minnesota

As students and teachers of mathematics, we are confronted all the time by the question: What is it all about? What are we doing it for? Are we accomplishing something of which we can give an intelligible account? Or are we merely doing what our predecessors did because somebody else had done it before them? Are we, like Christopher Robin,[2]

". . . . following an elephant
That's following an elephant
That's following an elephant
That isn't really there"?

I do not believe we are in this unprofitable state. If I did, I should not have taken this occasion to say so. But it is no confession of weakness to recognize that the problem is always with us. In its nature it is one for each generation that comes along. We are not asking whether it is important that mathematical knowledge should exist and be on record. That is admitted without question. There are tons of mathematical books in our libraries, and we accept the fact of their existence without uneasiness. The question is why we and our pupils should make it a chief concern of our lives to study those books, and look beyond the printed pages for the meaning that is behind them, and try to do the things that the authors have left undone.

I do not pretend to offer a final and compelling answer, resolving all doubts and removing all difficulties. If I had such an answer, I should not have been so unmindful of my obligation to society as to keep it a secret all this time. But as years pass I do find certain convictions increasingly confirmed in my own mind.

The question must be discussed with reference to the circumstances of the people on whose behalf it is asked. A reason that was good under the conditions of twenty-five years ago may not be a good one now, and if it is, that is a new observation, and not a mere reiteration of an old one.

On the other hand, regarding the problem as personal rather than universal, we can limit its scope, and so escape being carried too far afield. I have no intention of arguing the value of mathematical instruction in the public schools, or the importance of the Pythagorean theorem to the man in the street. I am thinking specifically of the members of the Mathematical Association of America, and of the organizations affiliated with it, a few thousand individuals

[1] Address of the retiring president of the Mathematical Association of America, at Nashville, Tenn., December 29, 1927.

[2] The writer offers his apologies to Christopher Robin for a slight adaptation of the latter's phraseology.

at most in a population of a hundred millions. Why should we, few as we are, spend our time doing mathematics, when there is so much else to be done? What shall we tell our students, when they ask us what we are driving at?

With our own peace of mind is involved the effectiveness of our instruction. We are told constantly that motivation is a prime requisite in teaching. Whatever that may imply for the technique of elementary instruction, the college student is a person of some maturity, and does not require motivation hour by hour and page by page. If it is available, so much the better. A young instructor in perplexity came to my colleague Professor Brooke, and inquired: "What do you do when students ask you what this or that particular topic in the calculus is good for in engineering?" To which Professor Brooke replied: "I tell them." But generally speaking, our students have some continuity of purpose, to the extent that they have purpose at all, and the most effective motivation is confidence that the instructor knows what he is about.

There was a time when you could make a good beginning by a reference to a liberal education. We have to recognize now that the thing formerly called by that name no longer exists. There is no body of knowledge or type of intellectual experience which is the common possession of persons of academic training, a bond of fraternity and medium of understanding between one educated man and another. It is possible to be influential in the intellectual life of the community without ever having opened a Latin grammar or a trigonometry. It is possible, as I happen to know from a specific instance, to graduate from an American university and think that Columbus discovered America in 1642. The traditional education survives as an experience of individuals, but not as a badge of a coherent and dominant group in society.

I do not admit for a moment that this is a sign of retrogression or decay. It is an incident of progress into the unknown. There is so much more to be learned that there has to be a division of labor in the learning of it; and there are thousands of people who have never been educated in the old-fashioned way, but whose intellectual attainments the alumnus of the old-fashioned education is compelled to respect. We have achieved democracy, or it has been achieved for us, if not in perfection, at any rate to a degree never contemplated by those who invoked it. And now that we have it, we have the problem of discovering what to do with it. Incidentally, we can not pretend that its destinies are committed to the hands of a class characterized by an ability to solve quadratic equations.

Another quaint phrase which we have heard from the lips of old people is "academic leisure." According to the theory of academic leisure, a professor taught his classes for an hour or two, and then had nothing more to do for the rest of the day. He retired to his desk to study undisturbed, or relaxed in intellectual companionship with students who chose to profit by his counsel.

Nowadays, if a professor has leisure, it is either because his colleagues have discovered by sad experience that he is incompetent outside his own limited sphere, or because he is protectively endowed by nature like the animal of which Hilaire Belloc writes:

"I shoot the hippopotamus with bullets made of platinum,
Because if I use leaden ones his hide is sure to flatten 'em."

There are so many things to be done, things that really matter to somebody, things that are demanded every day and every hour with an urgency well-nigh irresistible! While as for any particular piece of research, one realizes that no hungry world is clamoring for it, and if by an infrequent chance it really is relatively important, it will be done independently by two or three other people in the course of the next few years. The fact that there is a personal satisfaction in doing it, does not meet the issue. The investigator who used to boast that his researches amused only himself, and were of no possible interest to anybody else, may still investigate, but he does not boast. When the gentleman of leisure was the ideal of society, a gentleman whose activities were harmless was a good citizen. But this democracy which has happened so unexpectedly demands that you do something useful, or at least make out a case for the usefulness of what you do. More than that, if you are not satisfied with evasion, it demands a degree of usefulness at least comparable with the alternatives that press for attention. To those of us who have to earn a living, and through force of circumstances would not be gentlemen of leisure even if that profession still enjoyed its old prestige, there is significance in Gottlieb's words in "Arrowsmith": "Why should the world pay me for doing what I want, and what they do not want?" The relentless question for the individual is not why he should do mathematics, but how he can persistently refuse to do a hundred other things that leave no time or energy for mathematics.

Well, in the first place, I am not abashed to say that I believe that mathematical training has a great deal to do with thinking straight. The force of this contention has been so much exaggerated on the one hand, and so much depreciated on the other, that stress on its residual merit is not out of place. Let it be granted that prejudice and error have been known to survive exposure to courses in geometry. It is no less true that malignant organisms in the body frequently survive exposure to the most approved medical treatment. Let it be granted that the decisions of every-day life are seldom reached by the formal application of logic, and that those who know how to think logically need constantly to be on their guard not to let their reasoning faculties corrupt their better judgment. Still it is of inestimable benefit to any nation to have a few men within its borders who know the difference between a sound argument and a specious one, and to have abroad in its population a wide-spread consciousness that there are norms of thought transcending factional expediency. It

is my firm conviction that our average in this respect is higher for the presence among us of those of our colleagues and pupils, however few they may be, who consciously look for the wider significance of the standards with which their mathematical experience has familiarized them.

Mathematics not only demands straight thinking; it grants the student the satisfaction of knowing when he is thinking straight. There is no other field in which he can so soon stand on his own feet and face the world with confidence, assured that on his own ground his authority is final. Not many of your students, to be sure, ever reach this stage. Most of them keep on running to you with each successive bit of work, to ask you if it is right. But now and then, with your encouragement and guidance, one of them comes of age, having learned the meaning of self-criticism without self-distrust, and takes his place among those who know whereof they speak. He is potentially a man fit to exercise authority, with a full realization of the responsibility that goes with it.

A distinguishing characteristic of mathematics is its universality, its independence of time, place, and circumstance. The philosophers have made a great deal of this. They have ascribed to mathematical truth an absolute quality, transcending human experience and human existence. Perhaps they are right; nobody can prove the contrary. The alleged absoluteness of mathematics has always made it a refuge for souls who wanted to shut themselves away from the world. Probably the science always will owe much to those who see in it an escape from human limitations. But the universal persistence of mathematics has an appeal also for those of us who can not grasp so profound a metaphysical formulation. Even if we cannot attain anything beyond human experience, there is a compelling power in processes of thought which we know we have in common with intelligent men of all races and all times. Throughout the bewildering diversity of human nature, we have in the recognition of the facts of mathematics something as dependable and predictable as any phenomenon in the physical universe. By the exercise of our mathematical faculties we are brought into community of experience with Thales and Newton and their every-day associates, and their successors whose great-grandfathers have not yet been born. To this extent, the universal brotherhood of man is not a hypothesis or an aspiration, but an intimately accessible fact.

Still another phrase that needs to be rescued from abuse and restored to its proper place of dignity is "the enlargement of human knowledge." The words cover, if not a multitude of sins, at any rate a multitude of futilities. I may take this occasion to report, as a result of my own research (executed by a few seconds' manipulation of a calculating machine), that the product of the numbers 63471928 and 84023963 is 5333162929810664. I am confident that this fact is not to be found in the literature, and that it was never known until I

discovered it. If there is exalted merit in such an addition to knowledge, I am ready to propose an unlimited number of topics for investigation.

But the point that I really want to make is that this *reductio ad absurdum* is as far from being final as the shibboleth against which it is directed. Collectively and in the long run we are finding out things that the world wants to know, however ephemeral most of our individual contributions may be. To mention specific instances, the methods in exterior ballistics developed by Moulton and Bliss during the war depend on the Picard process of successive approximations and the introduction of an adjoint system of differential equations, familiar products of modern mathematical research. The preliminary work on which the theory of relativity is based was as much mathematical as physical. The quantitative relations which make the theory of statistics forbiddingly abstruse to the non-mathematician are commonplaces in the study of orthogonal functions and approximation by means of them. While the inventors of the statistical measures in question do not appear to have been extensively familiar with this particular part of the mathematical background, and while their achievement is all the more remarkable on that account, the general theory is available for the enlightenment of the student.

More fundamentally, the calculus is an essential part (one hesitates to say an integral part) of our civilization, and it is the research of the last hundred years that has given us a real understanding of the meaning of limits, and assured us that the processes of the calculus will bear critical examination. I believe that in fifty years more we shall have achieved a similar clarification of the concepts of probability. As in the case of the notion of limit, which can be defined in thirty seconds, but can be really appreciated, even by gifted graduate students, only after months of study, and took decades in reaching general acceptance, it will not be a question merely of a few formulations and theorems, but of years of accumulated experience, the gradual suppression of what is irrelevant and the emergence of what is essential. Incidentally, in going about our business toward the attainment of this end, we shall not only be offering the products of our own study, but shall be bringing together the biologist, the psychologist, the economist, and the physician, and making available to each the common elements in the experience of all.

No less important perhaps than what we contribute to the technique of the natural sciences is what we learn directly about the workings of our own minds. When the attempt has been made to analyze the processes of rational thought, the case of mathematical reasoning has always been cited as typical. Our activity is a continuing laboratory experiment, which is no nearer finality than the rest of the science of psychology, and I believe there is much still to be learned from it. We gradually recognize, for example, that the achievement of a mathematical proof in our consciousness is an experience rather than an

act, and that clarification is as likely to accompany relaxation following effort as to result immediately from the effort itself; but we can only guess how far these observations may ultimately be of theoretical and practical importance. Again, in trying to trace the inheritance of mental traits, we are likely to come down to specific cases by looking for something that can be identified as mathematical ability. One does not expect that the relations will be direct and simple. On finding that the highest score in an examination in my calculus class was made by a sister of a girl who had been similarly distinguished in the corresponding class a year or two earlier, and being informed that a third sister was taking freshman mathematics, I inquired as to her progress there, and was told that she wasn't doing so well in trigonometry, but got a B+ in English for a theme about trigonometry as the course that she disliked most. But however individual manifestations vary, it may turn out that there are fundamental differences in the chemical constitution of the nervous system which make sustained intellectual concentration practicable for one person, and insuperably distasteful to another. And whatever the answer may be, we are as likely as anybody to have something to offer toward the solution.

On the plane of social rather than individual psychology, there is a fascinating subject of inquiry in the relation between mathematical advances and the general consciousness of the age that produces them. Is there any underlying reason why Newton should have been a product of the age following Shakespeare, and Gauss a contemporary of Beethoven? Conversely, the main currents of mathematical progress must continue to have a profound influence on the habits of thought of the race. Who can estimate how much the emancipation of the intellect owes to the putting of the circle-squarer in his proper place, and the evidence of Gauss, Bolyai, Lobachewsky, and Riemann that the authority of Euclid did not finally seal the book of knowledge? The modern mathematician, like his predecessors, has done his part for the liberation of the human spirit.

In conclusion, let me emphasize that the advancement of mathematics is a great human enterprise in which we are all participants, and not mere spectators. You are making a contribution if you carry through a piece of research, or if you accept a committee appointment which leaves a colleague free to carry on his research, or if you study what he has done and give him the benefit of your appreciation and criticism, or if you merely work over the existing material and add to the fund of experience from which comes the collective assurance that we are on the right track. The science shares the vitality of all men in whose consciousness it exists.

A GEOMETRICAL PROBLEM IN WHICH ORDER OF CHOICE IS IMPORTANT

By W. C. GRAUSTEIN, Harvard University

Order of choice is frequently an essential element in analysis. It is only rarely, however, that it occurs in geometry. Recently the author encountered an instance of it which is of unusual interest because of the importance of the geometrical question at issue.

The question is simple to formulate. Can a collineation between two distinct planes in space always be established by a single projection? In other words: if a projective correspondence between the two planes is given, is it always possible to move the planes rigidly into new positions in which the lines joining corresponding points go through a point or are parallel?

The question, thus formulated—in the usual way, is enigmatic. It cannot be answered categorically with either "yes" or "no." It pertains to projective geometry and yet it is not purely projective in that a metric geometry must either be at hand or be introduced in the space in order to make rigid motions possible. From a historical point of view, it belongs to the hybrid class of questions originating during the period in which projective geometry was not as yet divorced from metric elements. The answer to it depends on whether it is considered from this essentially metric point of view or from the point of view of pure projective geometry.

From the metric point of view we start with an extended Euclidean space, i.e., with a projective space in which a definite Euclidean geometry has been introduced, and consider in this space an arbitrary projective correspondence between two arbitrarily chosen distinct finite planes. Our question must then be answered in the negative.

From a purely projective point of view, the point of departure is a projective space in which is given the projective correspondence between the two planes. Here the metric is introduced into the space *subsequent* to the projective correspondence, and it is always possible to choose it so that our question can be answered affirmatively.

In establishing these statements it will be necessary to recall briefly the salient metric properties of collineations.[1]

Metric point of view. In this case our space is metric from the start, and we choose in it arbitrarily a collineation between two finite planes, M and M'. It is convenient to distinguish three cases, according as the collineation is non-affine, affine but not conformal, or a transformation of similarity.

The only projections in metric space which establish non-affine collineations

[1] For an exhaustive treatment of these properties, the reader is referred to Sturm, *Die Lehre von den geometrischen Verwandtschaften*, II, §§40, 42.

are the central projections of one of two intersecting planes on the other. Hence, a necessary condition that it be possible to establish a given non-affine collineation between the planes M and M' by a single projection is that there exist at least one finite line in M which is transformed as by a rigid motion. This condition is also sufficient. If L and L' are lines in M and M' which correspond so that distance is preserved, the plane M' can be moved rigidly so that L' coincides with L and every point of L' coincides with the corresponding point of L. If, then, A, A' and B, B' are two distinct pairs of corresponding points, not lying on L, the corresponding lines AB and $A'B'$ intersect on L and hence the lines AA' and BB' meet in a point Q. The projection from Q of M on M' establishes between M and M' the given collineation.

As a matter of fact, there are two distinct finite lines in M which are transformed as by a rigid motion by a non-affine collineation. If the non-affine collineation, expressed in terms of homogeneous metric coordinates in the two planes, has the equations

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j, \qquad (i = 1,2,3) \quad |a_{ij}| \neq 0,$$

they are the lines

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = kx_3,$$

parallel to the vanishing line, for which[2]

$$(a_{31}^2 + a_{32}^2)k^2 = (A_{13}^2 + A_{23}^2),$$

where A_{ij} is the cofactor of a_{ij} in $|a_{ij}|$.

THEOREM 1. *A non-affine collineation between the points of two distinct finite planes in metric space can be established by a central projection of one of two intersecting planes on the other. Conversely, every projection of this type establishes a non-affine collineation.*

If the given collineation is a transformation of similarity, the facts are obvious.

THEOREM 2. *A transformation of similarity of one of two distinct finite planes in metric space into the other can be established by a projection of one of two parallel planes on the other, and conversely.*

There remain, on the one hand, the collineations which are affine but not conformal, and on the other hand, the parallel projections of one of two intersecting planes on the other. It is true that a parallel projection which relates to one another two intersecting planes establishes a non-conformal affine trans-

[2] Since we are restricting ourselves to the geometry of reals, neither $a_{31}^2+a_{32}^2$ nor $A_{13}^2+A_{23}^2$ can vanish.

formation. But it is *not* true that every non-conformal affine transformation can be established by a single projection. Herein lies the reason for the fact that when our question is considered from the metric point of view, it must be answered in the negative.

It is evident that a necessary, as well as sufficient, condition that it be possible to establish a given non-conformal affine transformation between M and M' by a single projection is that there exist at least one line in M which is transformed so that distance is preserved. It is, moreover, well known that an affine transformation multiplies by the same constant factor, k, the lengths of all line-segments in M with the same direction. Hence, if C is a *unit* circle in M and E is the ellipse in M' into which C is carried, the factor k by which all distances in the direction of a radius of C are multiplied is precisely the length of the semi-diameter of E which corresponds to the radius of C. Thus, k varies from the semi-minor axis b to the semi-major axis a of the ellipse E:

$$b \leqq k \leqq a,$$

and therefore takes on the value 1 if and only if 1 lies in the closed interval from b to a.

THEOREM 3. *There are two, one, or no pencils of real parallel lines which are transformed by a non-conformal affine transformation so that distance is preserved, according as unity lies within, is an end point of, or lies without the interval from b to a, where b and a are the semi-axes of an ellipse which corresponds to a unit circle by the transformation.*

For the transformation

$$x' = a_1x + b_1y + c_1, \qquad y' = a_2x + b_2y + c_2, \qquad a_1b_2 - a_2b_1 \neq 0,$$

the value of k for the direction of slope-angle θ is found to be

$$k^2 = (a_1^2 + a_2^2)\cos^2\theta + 2(a_1b_1 + a_2b_2)\cos\theta\sin\theta + (b_1^2 + b_2^2)\sin^2\theta.$$

It is evident from this expression, and also geometrically, that non-conformal affine transformations actually exist for which k never takes on the value unity.

Projective point of view. Since, here, our space is in the beginning projective, and metric geometry is not introduced until after the projective transformation of M into M' is chosen, we are free to adapt the metric geometry to the given transformation. In particular, we take as the fixed plane—the plane at infinity—of the metric geometry a plane which meets M and M' in lines which do *not* correspond by the given transformation. These lines become, then, the lines at infinity in M and M', and the lines to which they correspond become vanishing lines. Consequently, the given projective transformation of M into M', considered with reference to the resulting metric space, is a non-

affine collineation. But, by Theorem 1, a non-affine collineation between M and M' can be established by a single projection.

THEOREM 4. *If a projective correspondence between the points of two distinct planes M and M' in projective space is given, and there is introduced in the space a metric geometry referred to a properly chosen fixed plane, then the planes M and M' can be moved rigidly into new positions in which the lines joining corresponding points go through a point.*

STEREOGRAPHIC PROJECTION OF A QUADRIC

By CHARLES A. RUPP, Pennsylvania State College

1. Introduction. By projecting plane configurations stereographically onto a quadric, we obtain a number of interesting configurations which deserve to be better known. Thus we rediscover the theorem of Hesse[1] that a Pascal hexagon upon a quadric is likewise a Brianchon hexagon, and readily find several other interesting theorems. This paper aims to show the ease with which theorems are transferred from the plane to a quadric, and to give independent proofs of some of the outstanding theorems thus attained.

Klein[2] was one of the first to use stereographic projection as a means of finding new theorems, and Segre,[3] in his laureate thesis, suggests that the method is apt to be fruitful in a general study of configurations upon hyperquadrics. The present paper is a part of such a general study, but is restricted to non-degenerate quadrics in S_3.

2. Coordinate Machinery. We shall adopt the notation of Bertini[4] for the important spaces in the projection. For the center of the stereographic projection take a point P on the quadric ϕ, and project the points of the quadric upon a plane π which does not pass through the point P. We denote by τ the tangent plane to the quadric ϕ at the point P, and by σ the line of intersection of the planes π and τ. The intersection of ϕ and its tangent plane τ is the pair of rulings through the point P, and is a degenerate conic called ϕ'. The line σ also meets ϕ in a degenerate conic, called ϕ'', which is the pair of points A and B. We have given names to three quadratic spaces, ϕ, ϕ', and ϕ'', and to three linear spaces, π, τ, and σ. We shall see their significance as soon as we have set up the equations of the projection.

Let the coordinates of the points P, A, and B be respectively (0, 0, 0, 1),

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[1] Hesse, *Werke*, p. 58; p. 651; p. 676.

[2] Klein, *Ueber einen liniengeometrischen Satz*, Mathematische Annalen, vol. 22 (1883), p. 325.

[3] Segre, *Studio sulle quadriche*, Mem. dell'Acad. di Torino, vol. 36 (1884).

[4] Bertini, *Proj. Geom. Mehrdimensionaler Räume*, Chapter VI, Section 16.

(1, 0, 0, 0), (0, 0, 1, 0), and for the fourth vertex of the coordinate tetrahedron choose the point C common to the second rulings of ϕ which go through A and B. This tetrahedron is both inscribed in and circumscribed to the quadric ϕ. By a proper choice of the unit point, we can reduce the equation of ϕ to the form $x_0x_2 - x_1x_3 = 0$. The equations of the linear spaces π, τ, and σ are respectively

$$x_3 = 0, \quad x_1 = 0, \qquad x_2 = x_1 = 0.$$

Recalling that ϕ' is the intersection of ϕ and τ, we have

$$\left.\begin{aligned} x_0x_2 - x_1x_3 &= 0 \\ x_1 &= 0 \end{aligned}\right\} \text{ or } \begin{cases} x_1 = x_2 = 0 & (PA) \\ x_1 = x_0 = 0 & (PB). \end{cases}$$

ϕ'' was defined as the intersection of ϕ and σ; hence

$$\left.\begin{aligned} x_0x_2 - x_1x_3 &= 0 \\ x_1 = x_3 &= 0 \end{aligned}\right\} \text{ or } \begin{cases} x_1 = x_3 = x_2 = 0 & (A) \\ x_1 = x_3 = x_0 = 0 & (B). \end{cases}$$

3. Equations of Projection. The projection going from ϕ to π is straightforward. A point M of ϕ of coordinates (y_0, y_1, y_2, y_3) has for its image a point M' of π of coordinates $(y_0, y_1, y_2, 0)$. It is on the return trip from π to ϕ that we first encounter irregularities. The transform of a point M' of π of coordinates $(x, y, z, 0)$ is found by getting the piercing points on ϕ of the line PM', one of which is P and the other is the true transform, the point M of ϕ. We find

$$x_0 : x_1 : x_2 : x_3 = yx : y^2 : yz \cdot xz$$

which are the equations of the stereographic projection.

If $y=0$, we get only the point P, no matter what point on the line we take. This line is the line σ, and is a fundamental line for the projection, its transform being the fundamental point P. If, besides $y=0$, we also have x or $z=0$, the transformation becomes indeterminate. This means that the points A and B are fundamental points in the plane π. Let us examine more closely what happens to A.

Consider a line S_1' of π through A, of equation $y=\lambda z$. The transform of a point on this line has coordinates

$$\lambda zx : \lambda^2 z^2 : \lambda z^2 : xz = \lambda x : \lambda^2 z : \lambda z : x.$$

From this we see that as the point M' moves up the line S_1' towards A, the transform on ϕ approaches the point of coordinates $(\lambda, 0, 0, 1)$, which is a definite point on the line $x_1=x_2=0$, or PA. Thus the fundamental point A is transformed into the fundamental line PA, and in such a way that we attain a definite point of PA as we approach A in a definite direction. In like manner the transform of the fundamental point B is the fundamental line PB. Thus

we have seen that the transform of the quadratic space ϕ'' is the quadratic space ϕ'.

It is readily shown that a point on the fundamental line σ comes from a tangent to the quadric ϕ at P, which the Italian geometers would express by saying that the transform of a point on σ is a point on ϕ "infinitesimally close" to P.

What is the transform of the line S_1' considered above? The equation of its projecting plane is $x_1-\lambda x_2=0$, which meets ϕ in a conic which breaks up into the line $x_1=x_2=0$, or PA, and the line S_1, which we may regard as the true transform of S_1', whose equations are $x_1-\lambda x_2=0$ and $x_0-\lambda x_3=0$. The transform of any line in π through one of the points of ϕ'' is a generator of the quadric ϕ. It readily follows, as Klein and Clebsch have pointed out, that the net of rulings on ϕ is projected into the net of the two pencils of lines of π whose vertices are at the fundamental points of the plane. Since there are but two of these fundamental points, there are but two families of lines upon the quadric. The well known properties of these families, that two rulings of the same family never meet, and that two rulings of different families always meet, follow at once from the properties of the two pencils in π.

It may be interesting to note here that in the more general theory of hyperspace, any S_m' of π which contains an S_{m-1} of ϕ' is the image of a linear space S_m of ϕ.

4. Theorems. The transform of the line joining a pair of points of π is a conic through P and the transforms of the points. A pair of lines of π which meet in a point M' are the images of a pair of conics on ϕ through P, which also meet at the point M. The transform of a triangle of π is a configuration upon ϕ which we may call a *triconic*. A triconic is a set of three conics on ϕ passing through the point P. The three conics are termed the sides of the triconic, and each pair has a point in common (besides P), which defines the three vertices of the triconic. In like manner, the transform of a hexagon of π is a set of six conics and six points of ϕ which we call a *hexaconic*. A conic of π will correspond to a quartic on ϕ which has a double point at P, whose nodal tangents will be fixed by the points where the fundamental line σ intersects the original conic of π. The image of a general conic on ϕ is a conic of π through the fundamental points A and B.

With these preliminaries behind, let us consider some of the classic configurations of the plane. Desargues's theorem on triangles says that two triangles which are perspective from a point are also perspective from a line, and conversely. When we project these two triangles up onto the quadric, we obtain

THEOREM 1. *If two triconics on ϕ are such that the intersections of corresponding conics lie on a conic through the point P, then the conics joining pairs of op-*

posite vertices meet in P and another point. There are a number of special cases of this theorem, arising from the various relations which the Desargues line may have to the fundamental points and line of π. We shall state one corollary that occurs when the Desargues line and σ coincide.

COROLLARY 1. *If two triconics on ϕ are such that corresponding sides share the same tangent at P, then the conics joining pairs of corresponding vertices meet in P and still another point.*

Pascal's theorem states that the necessary and sufficient condition that the vertices of a hexagon lie upon a conic is that the points of intersection of opposite sides be collinear. This leads to

THEOREM 2. *The necessary and sufficient condition that the vertices of a hexaconic lie upon a quartic on ϕ through P is that the points of intersection of opposite conics lie on a conic of ϕ through P.*

A special case of this theorem is when the hexaconic degenerates to a hexagon. Consider a [skew] hexagon upon ϕ, whose sides we call 1, 2, 3, 4, 5, 6. It will be convenient to number the vertices also, the scheme being that point 1 is the intersection of lines 1 and 2, $\cdots$, point 6 is the intersection of sides 6 and 1. Clearly lines 1, 3, and 5 belong to one family of rulings of ϕ, and the other sides to the other family. Since we plan to investigate this configuration in some detail, we take the rulings 1, 2, 4, and 5 to be the edges of the coordinate tetrahedron CA, AP, CB, and BP, and let the rulings 3 and 6 pass through the unit point, which we know to be on ϕ. We shall denote by primes the images of these points and lines, and we shall study the configuration we get by projection upon π to see if it gives us any information about the corresponding configuration on ϕ. It is this systematic use of stereographic projection, both to arrive at the knowledge of the existence of certain configurations of linear spaces upon a quadric, and to ascertain the properties of given configurations upon the quadric, which is the *raison d'être* of this paper.

When we project upon π the skew hexagon formed of six rulings of ϕ, we see that the points $1'$ and $2'$, the images of two vertices of the skew hexagon, coincide at the point A, and in like manner, $4'$ and $5'$ coincide at B. The image hexagon is accordingly degenerate. It is a Brianchon hexagon, for its sides are tangent to the conic ϕ'', and it is also a Pascal hexagon, for its vertices lie on the conic formed by the pair of lines CA and CB. Thus it has a Pascal line and a Brianchon point, and since this projected hexagon has shrunk to a complete quadrangle, it turns out that the Brianchon point is a vertex of the diagonal triangle, and the Pascal line is the opposite side. In the plane π, then, the Brianchon point and Pascal line are pole and polar with respect to both the Brianchon conic ϕ'' and the Pascal conic.

Let us now give a couple of definitions. A Pascal skew hexagon is one whose pairs of opposite planes meet in three coplanar lines, this plane being the Pascal plane of the hexagon. A Brianchon skew hexagon is one whose diagonals have a point in common. We may now return to the quadric and utilize the information we gleaned from affairs in π.

THEOREM 3. *A skew hexagon upon a quadric is both a Pascal and a Brianchon hexagon. The Brianchon point and the Pascal plane are polar with respect to the quadric.*

This theorem was discovered some eighty-five years ago by Hesse[1], whose attack, if we may judge by his proof, was purely synthetic. Since we have nearly all the material needed for a direct analytic proof, we go on to make one.

Upon the quadric ϕ of equation $x_0x_2-x_1x_3=0$ we have taken six generators, of equations

$$\begin{array}{llll} 1 & x_2 = x_3 = 0 & 2 & x_1 = x_2 = 0 \\ 3 & x_1 = x_2,\ x_0 = x_3 & 6 & x_0 = x_1,\ x_2 = x_3 \\ 5 & x_0 = x_1 = 0 & 4 & x_0 = x_3 = 0. \end{array}$$

These six rulings meet in nine points, whose coordinates are

$$\begin{array}{llllll} 1 & (1,0,0,0) & 2 & (1,0,0,1) & 3 & (0,1,1,0) \\ 4 & (0,0,1,0) & 5 & (0,0,1,1) & 6 & (1,1,0,0) \\ 7 & (0,0,0,1) & 8 & (0,1,0,0) & 9 & (1,1,1,1) \end{array}$$

where the first six points have already been defined as vertices of the hexagon, and the last three points are P, C, and U.

We proceed to verify that the hexagon has a Brianchon point. The pairs of opposite vertices are the points 1, 4; 2, 5; and 3, 6. The point whose coordinates are (1, 0,–1, 0) is clearly a linear combination of the coordinates of the points taken in pairs; hence this point is on the three diagonals, and is the Brianchon point of the hexagon. The pairs of opposite planes are the planes fixed by the pairs of lines 12 and 45. We shall designate these six planes of the hexagon by the lower number in the pair of sides defining them. For the lines of intersection,

$$\begin{cases} \text{Plane } 1 & x_2 = 0 \\ \text{“ } 4 & x_0 = 0 \end{cases} \quad \begin{cases} \text{Plane } 2 & x_1 - x_2 = 0 \\ \text{“ } 5 & x_1 - x_0 = 0 \end{cases} \quad \begin{cases} \text{Plane } 3 & x_0 - x_3 = 0 \\ \text{“ } 6 & x_2 - x_3 = 0, \end{cases}$$

which lines are precisely those fixed by the points 78, 97, and 89. It is obvious from the displayed equations that the plane $x_0=x_2$ contains all three lines, and hence is the Pascal plane.

[1] Hesse, loc. cit.

The polar plane of y with respect to ϕ has the equation

$$y_2x_0 - y_3x_1 + y_0x_2 - y_1x_3 = 0.$$

The polar plane of the point (1, 0, 1, 0) is $x_0-x_2=0$. This completes the analytic proof of the theorem.

We note in passing that the line joining opposite vertices of the skew hexagon is the conjugate of the line of intersection of opposite planes, which follows from the self-duality of the hexagon.

Since generators of the same family do not meet, we can arrange the six rulings into only six hexagons, not sixty. These six are obtained by holding three alternate sides fixed, and permuting the others. In the plane π we have, then, two conjugate Steiner points, which happen, as a glance at the figure of the complete quadrangle in π shows us, to be points already obtained, namely, the points of coordinates $(1, 0, -1)$ and $(1, 1, 1)$. The six Brianchon points are likewise arranged by threes, upon two lines whose equations are seen to be $x-y+z=0$ and $x-z=0$. We expect that there will be some relations between the Pascal planes and the Brianchon points of the skew hexagons, but we cannot go back directly from π because the points and planes in the space figure are not on ϕ. We shall have to find the coordinates of the Brianchon points and the Pascal planes directly from the knowledge of the sides of the hexagons.

The six possible hexagons have their sides arranged in the orders

$$\begin{array}{lll} 1\,2\,3\,4\,5\,6, & 1\,4\,3\,6\,5\,2, & 1\,6\,3\,2\,5\,4, \\ 1\,6\,3\,4\,5\,2, & 1\,2\,3\,6\,5\,4, & 1\,4\,3\,2\,5\,6, \end{array}$$

and their vertices are the points

$$\begin{array}{lll} 1\,2\,3\,4\,5\,6, & 8\,3\,9\,5\,7\,1, & 6\,9\,2\,7\,4\,8 \\ 6\,9\,3\,4\,7\,1, & 1\,2\,9\,5\,4\,8, & 8\,3\,2\,7\,5\,6. \end{array}$$

The six Brianchon points have the coordinates

$$\begin{array}{lll} (1,0,-1,0), & (0,1,1,1), & (1,1,0,1), \\ (1,1,1,0), & (1,0,1,1), & (0,1,0,-1), \end{array}$$

and the six Pascal planes have the equations

$$\begin{array}{lll} x_0 - x_2 = 0, & x_0 - x_1 - x_3 = 0, & x_1 - x_2 + x_3 = 0, \\ x_0 + x_2 - x_3 = 0, & x_0 - x_1 + x_2 = 0, & x_1 - x_3 = 0. \end{array}$$

The first three Brianchon points are seen to lie on a line whose equations are $x_0-x_1+x_2=x_1-x_3=0$, and this same line lies on each of the second trio of Pascal planes. The second triple of Brianchon points lies on the line whose equations are $x_0-x_2=x_1-x_2+x_3=0$, and through this line pass the first three Pascal planes. These lines are obviously conjugate with respect to ϕ. They may be

called Steiner lines, for they are common to the triples of Pascal planes. We may now collect our results in the form of

THEOREM 4. *If six rulings of a quadric form a skew hexagon, they form six hexagons, each of which has a Pascal plane and a Brianchon point which are pole and polar with respect to the quadric. The six Pascal planes pass by threes through two polar lines, and the six Brianchon points lie by threes on the same two lines.*

FAMILIES OF CURVES OF PURSUIT, AND THEIR ISOCHRONES

By ALFRED J. LOTKA, Statistical Bureau, Metropolitan Life Insurance Company

A family of curves of pursuit may be defined by the condition that all curves of the family shall meet the path of the pursued point in the same fixed point.

Let P_1 be the point pursued, which moves with a constant velocity v_1 and in a straight line P_1O. Let P_2 be the pursuing point, which moves always in the direction P_2P_1, with a constant velocity v_2. If $v_2 > v_1$, P_2 will ultimately catch up with P_1 at some fixed point O. We may speak of O as the point of capture.

Now we may ask, for a given ratio $v_1/v_2 = \epsilon$, and a given direction of v_1, what is the locus of all those *positions at time t* of the pursuing point, which will result in capture at the given point O?

The problem can be solved by applying the equation of curves of pursuit and expressing the constants of integration in terms of the initial configuration and velocity components. This was the method originally followed by the writer. A briefer and more direct method communicated to him by Professor F. R. Sharpe is the following:

Denoting by x, y, the rectangular coördinates of P_2, with the origin at O and the x axis along OP_1, and denoting by s the length of the curve of pursuit from O to P_2, we have, evidently,

$$x = \epsilon s - \rho \cos \phi \qquad y = \rho \sin \phi$$

where ϕ is the angle P_2P_1O, and ρ is the radius vector P_1P_2. Differentiating we have:

$$(dx/ds) = \epsilon - (d\rho/ds) \cos \phi + \rho \sin \phi (d\phi/ds) = -\cos \phi,$$

$$(dy/ds) = \qquad (d\rho/ds) \sin \phi + \rho \cos \phi (d\phi/ds) = \sin \phi.$$

Solving we have

$$(d\rho/ds) = 1 + \epsilon \cos \phi, \qquad \rho(d\phi/ds) = -\epsilon \sin \phi.$$

Hence

$$(1 - \epsilon \cos \phi)(d\rho/ds) + \epsilon\rho \sin \phi (d\phi/ds) = 1 - \epsilon^2.$$

Integrating,

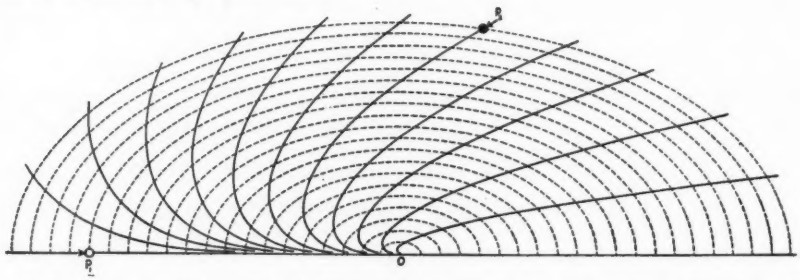
$$(1 - \epsilon \cos \phi)\rho = (1 - \epsilon^2)s \quad \text{and} \quad \epsilon s = \frac{\rho\epsilon}{1 - \epsilon^2}(1 - \epsilon \cos \phi) = D.$$

For a given value of D, and regarding ρ, ϕ, as polar coördinates of the point P_2, this represents an ellipse of eccentricity ϵ or v_1/v_2, and with its center at O and focus at P_1, a distance D from O. But the pursuing point, which at time t is at P_2, will ultimately meet the pursued point at O. Evidently this will take place at time $t+(D/v_1)$. In other words, points which at time t lie on the ellipse defined as above, will, after a lapse of time D/v_1, all simultaneously converge in the fixed point O. Or, conversely, all points on curves of pursuit of the character defined, which terminate at the fixed point O, lie, *at time t*, on an ellipse of eccentricity ϵ, with center at O, and one focus at the pursued point P_1. Thus the ellipses so defined are the loci of positions of equal time on the family of curves of pursuit. We may term these loci *isochrones*.

If we denote by the term *centrocline* the locus of all points in a family of curves, at which the slopes of the curves pass through a single point or *center*, it is clear that the *isochrone* ellipses are also the *centroclines* of the family of curves described, the corresponding *center* being that focus of the ellipse, at which the moving point is located.

This suggests a graphic method of mapping the family of curves of pursuit, analogous to the mapping of integral curves by means of isoclines.

A map of the family of curves of pursuit so plotted is shown on the accompanying diagram. The process hardly needs description. It may be remarked, however, that with a little resourcefulness in manipulation (using a loop of thread—fishing line is best because "inextensible"—and four pins, two at the focus, and two to establish "setting" points) a complete map of many curves can be drawn in less time that would be required to plot a single curve of pursuit by a point-by-point method.



Point P_2 moving with constant velocity v_2 and starting anywhere on outer ellipse, "pursues" point P_1 which starts at one of the foci, with constant velocity v_1. Capture takes place always at the same point O, the center of the ellipse. The concentric ellipses are *isochrones*, the curves of pursuit their *isoclinal trajectories*.

The following theorems are obviously implied in the developments set forth above, or follow by simple deduction:

THEOREM: *The isochrones of a family of curves of pursuit defined by a fixed point of capture, a pursued point moving with constant rectilinear velocity v_1 and a pursuing point moving with constant velocity v_2 directed always toward the pursued point,* are ellipses of eccentricity v_1/v_2, *with center at the fixed point of capture, and with one focus at the pursued point.*

THEOREM: *The isochrones of the family of curves specified are also its centroclines.*

THEOREM: *If the center of an ellipse of eccentricity $v_1/v_2 = \epsilon$ remains fixed while one focus approaches this center at a constant rectilinear velocity v_1, and if at the same time each point of the ellipse "pursues" the moving focus at a constant velocity v_2, then the ellipse, while shrinking continually, retains its eccentricity v_1/v_2 until it has completely shrunk into a point at the fixed center.*

THEOREM: *The foco-clinal trajectories*[1] *of a family of concentric ellipses of equal eccentricity make everywhere equal intercepts between two such ellipses.* (This theorem should be visualised by inspection of the figure, preferably with a pair of compasses handy. Where the curves of pursuit are relatively flat, the compasses give immediate evidence of the truth of the theorem.)

The case here discussed has a certain concrete background. We may view the matter in the following light: An organism P_1, of a species which serves as food to an organism P_2, becomes aware of the presence of the latter when at a distance ρ from the same, and immediately flees in a straight line to the nearest cover or refuge, distance D away. If at the moment of sighting, the polar coordinates ρ, ϕ of P_2 are such that

$$\rho > D(1 - \epsilon^2)/\epsilon(1 - \epsilon \cos \phi),$$

then the fleeing organism will escape, reaching cover before its pursuer. If on the other hand

$$\rho < D(1 - \epsilon^2)/\epsilon(1 - \epsilon \cos \phi),$$

then capture will occur. In other words, capture will or will not occur, according as the organism P_2 is inside or outside the ellipse

[1] This term seems self-explanatory. The curves of pursuit form a family which cross the ellipses everywhere at such an angle that the tangent at any point to a curve of pursuit passes through the focus of the ellipse crossed at that point. Thus the curves of pursuit are trajectories of the ellipses, inclined, at each crossing point, towards the focus.

$$\rho = D(1 - \epsilon^2)/\epsilon(1 - \epsilon \cos \phi)$$

at the time it is sighted by the pursued organism.

The present communication is of the nature of a preliminary step toward a more complete analysis of certain biological problems on which work is in progress.

AN EXPRESSION FOR THE SUMMATION $\sum_{m=1}^{n} m^p$

By R. S. UNDERWOOD, Texas Technological College

1. In this paper I shall derive by induction an expression for the summation $\sum_{m=1}^{n} m^p$ as a polynomial in n, whose coefficients incidentally define the Bernoulli numbers in the beautifully symmetric quadratic recursion relation:

$$(2r + 3)\frac{B_{2r+2}}{(2r + 2)!} = \frac{B_2 B_{2r}}{2!(2r)!} + \frac{B_4 B_{2r-2}}{4!(2r - 2)!} \cdots + \frac{B_{2r} B_2}{(2r)!2!}.$$

This equation was thought to be possibly new by others who looked over my work as well as by myself until I located it in slightly different form in N. Nielsen's *Nombres de Bernoulli*. However, the method by which I arrived at this result is totally unlike all other treatments of the Bernoulli numbers I have found, and the constants k_r which appear naturally by this method are of basic interest in themselves, since the equation connecting them [see (2) below] is stripped even of the factorials which appear apparently in all Bernoulli number recursion relations.

THEOREM; *Let* $S^{(p)} = \sum_{m=1}^{n} m^p = 1^p + 2^p + \cdots + n^p$ (p *an integer*), *and let*

$$(1) \quad \begin{bmatrix} 1 \\ p \end{bmatrix} = p(p - 1) = \frac{p!}{(p - 2)!}; \quad \begin{bmatrix} r \\ p \end{bmatrix} = p(p - 1) \cdots (p - r) = \frac{p!}{(p - r - 1)!}.$$

Then

$$(A) \qquad S^{(p)} = \frac{1}{p + 1} n^{p+1} + \frac{1}{2} n^p + k_1 p n^{p-1} - k_3 \begin{bmatrix} 2 \\ p \end{bmatrix} n^{p-3} + k_5 \begin{bmatrix} 4 \\ p \end{bmatrix} n^{p-5} - \cdots,$$

(*ending with the* n^2 *or* n *term*), *where the constants* k_r (r *odd*) *can be evaluated successively as follows*:

$$k_1 = \frac{1}{12}\,;\; k_3 = \frac{1}{3+2}(k_1k_1) = \frac{1}{6!}\,;\; k_5 = \frac{1}{5+2}(k_1k_3 + k_3k_1) = \frac{4}{3\cdot 8!}\,;\cdots$$

$$k_r = \frac{1}{r+2}(k_1\,k_{r-2} + k_3k_{r-4} + \ldots + k_{r-2}k_1)\,; \tag{2}$$

or, for quicker computation where r is large,

$$k_r = \frac{2}{r+2}\left(k_1k_{r-2} + k_3k_{r-4} + \cdots + k_{(r-5)/2}k_{(r+3)/2} + \frac{1}{2}k^2{}_{(r-1)/2}\right), \tag{3}$$

when $r = 4n-1$;

$$k_r = \frac{2}{r+2}\left(k_1k_{r-2} + k_3k_{r-4} + \cdots + k_{(r-3)/2}k_{(r+1)/2}\right), \quad \text{when } r = 4n+1.$$

The first nine values of k_r are

$$\frac{1}{12}, \frac{1}{6!}, \frac{4}{3\cdot 8!}, \frac{3}{10!}, \frac{10}{12!}, \frac{691}{15\cdot 14!}, \frac{280}{16!}, \frac{10,851}{5\cdot 18!}, \text{ and } \frac{438,670}{21\cdot 20!}.$$

These first nine constants enable one to write immediately the sums $S^{(p)}$ for every value of p up to and including 18. For instance,

$$S^{(18)} = \frac{1}{19}n^{19} + \frac{1}{2}n^{18} + \frac{19}{12}n^{17} - \frac{2\cdot 17}{5}n^{15} + 34n^{13} - \frac{3\cdot 13\cdot 17}{5}n^{11} + \frac{5\cdot 13\cdot 17}{3}n^9$$

$$- \frac{2\cdot 17\cdot 691}{5\cdot 7}n^7 + 2\cdot 3\cdot 7\cdot 17n^5 - \frac{3617}{2\cdot 5}n^3 + \frac{43,867}{2\cdot 3\cdot 7\cdot 19}n$$

$$= \frac{1}{3990}[210n^{19} + 1,995n^{18} + 5,985n^{17} - 27,132n^{15} + 135,660n^{13}$$

$$- 529,074n^{11} + 1,469,650n^9 - 2,678,316n^7 + 2,848,860n^5$$

$$- 1,443,183n^3 + 219,335n].$$

PROOF: Formula (A) yields a known result for $p=1$. Assuming it true up to and including $p=p$, with $\left[\begin{smallmatrix} r \\ p \end{smallmatrix}\right]$ and k_r defined as in (1) and (2), we have

$$(A) \qquad S^{(p)} = \frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + k_1pn^{p-1} - k_3\left[\begin{matrix} 2 \\ p \end{matrix}\right]n^{p-3} + \cdots$$

$$\pm k_r\left[\begin{matrix} r-1 \\ p \end{matrix}\right]n^{p-r} \mp \cdots.$$

Then

$$
\begin{aligned}
S^{(p+1)} &= 1\cdot 1^p + 2\cdot 2^p + 3\cdot 3^p + \cdots + n\cdot n^p \\
&= nS^{(p)} - \left[1^p + (1^p + 2^p) + (1^p + 2^p + 3^p) + \cdots \right. \\
&\quad \left. + \left\{1^p + 2^p + \cdots + (n-1)^p\right\}\right] \\
&= nS^{(p)} - \left(\frac{1}{p+1}1^{p+1} + \frac{1}{2}1^p + k_1 p 1^{p-1} - \cdots\right) \\
&\quad + \left(\frac{1}{p+1}2^{p+1} + \frac{1}{2}2^p + k_1 p 2^{p-1} - \cdots\right) \\
&\quad + \cdots + \left\{\frac{1}{p+1}(n-1)^{p+1} + \frac{1}{2}(n-1)^p + k_1 p (n-1)^{p-1} - \cdots\right\}\Bigg] \\
&= nS^{(p)} - \left[\frac{1}{p+1}(S^{(p+1)} - n^{p+1}) + \frac{1}{2}(S^{(p)} - n^p)\right. \\
&\quad \left. + k_1 p (S^{(p-1)} - n^{p-1}) - \cdots\right] \\
&= nS^{(p)} - \left[\left(\frac{1}{p+1}S^{(p+1)} + \frac{1}{2}S^{(p)} + k_1 p S^{(p-1)} - \cdots\right) - S^{(p)}\right].
\end{aligned}
$$

$$
\frac{p+2}{p+1}S^{(p+1)} = \frac{2n+1}{2}S^{(p)} - k_1 p S^{(p-1)} + k_3\begin{bmatrix} 2 \\ p \end{bmatrix}S^{(p-3)} - \cdots \mp k_r\begin{bmatrix} r-1 \\ p \end{bmatrix}S^{(p-r)} \pm \cdots
$$

$$
S^{(p+1)} = \frac{p+1}{p+2}P(n),
$$

where $P(n)$ is a polynomial in n.

Making the further definition that $\left[\begin{smallmatrix} j \\ p_i \end{smallmatrix}\right] = (p-i)(p-i-1)\cdots(p-j)$ (i, j integers and $i \leqq j$), we may arrange $P(n)$ in columns of like powers of n as follows:

n^{p+2}	n^{p+1}	n^p	n^{p-1}	n^{p-3}	n^{p-5}	n^{p-r+3}	n^{p-r+2}	n^{p-r+1}	n^{p-r}	
$\frac{1}{p+1}$	$+\frac{1}{2}$	$+k_1 p$		$-k_3\left[\begin{smallmatrix} 2 \\ p \end{smallmatrix}\right]$		$+\cdots\cdots$	$\cdot$	$\pm k_r\left[\begin{smallmatrix} r-1 \\ p \end{smallmatrix}\right]$		$\mp\cdots$
	$+\frac{1}{2(p+1)}$	$+\frac{1}{4}$	$+\frac{k_1 p}{2}$		$-\frac{k_3}{2}\left[\begin{smallmatrix} 2 \\ p \end{smallmatrix}\right]$	$+\cdots\mp\frac{k_{r-3}}{2}\left[\begin{smallmatrix} r-3 \\ p \end{smallmatrix}\right]$			$\pm\frac{k_r}{2}\left[\begin{smallmatrix} r-1 \\ p \end{smallmatrix}\right]$	$\cdots$
$+k_1 p$		$-\frac{1}{p}$	$-\frac{1}{2}$	$-k_1\left[\begin{smallmatrix} 1 \\ p_1 \end{smallmatrix}\right]$		$+\cdots\cdots$	$\cdot$	$\pm k_{r-2}\left[\begin{smallmatrix} r-2 \\ p_1 \end{smallmatrix}\right]$		$\mp\cdots$
$+k_3\left[\begin{smallmatrix} 2 \\ p \end{smallmatrix}\right]$				$+\frac{1}{p-2}$	$\frac{1}{2}$	$+\cdots\cdots$	$\cdot$	$\pm k_{r-4}\left[\begin{smallmatrix} r-2 \\ p_3 \end{smallmatrix}\right]$		$\mp\cdots$
$+k_5\left[\begin{smallmatrix} 4 \\ p \end{smallmatrix}\right]$					$-\frac{1}{p-4}-\cdot$		$\cdot$	$\pm k_{r-6}\left[\begin{smallmatrix} r-2 \\ p_5 \end{smallmatrix}\right]$		$\mp\cdots$
$\cdots\cdots$						$\cdots\cdots$				
$\cdots\cdots$						$\cdots\cdots$				
$+k_{r-3}\left[\begin{smallmatrix} r-3 \\ p \end{smallmatrix}\right]$						$\pm\frac{1}{p-r+3}\pm\frac{1}{2}$		$\pm k_1\left[\begin{smallmatrix} r-2 \\ p_{r-3} \end{smallmatrix}\right]$		$\mp\cdots$
$+k_r\left[\begin{smallmatrix} r-1 \\ p \end{smallmatrix}\right]$								$\mp\frac{1}{p-r+1}$	$\mp\frac{1}{2}$	$\mp\cdots$
$\vdots$										$\mp\cdots$

It will be noted that there is one positive and one equal negative term in n^{p-m} for every odd m; hence these terms disappear, and the exponents of all terms of $S^{(p+1)}$ except the second are even or odd (the last terms ending in n^2 or n) according as p is even or odd.

The coefficient of $n^{(p-r+1)}$ is

$$\pm \frac{p+1}{p+2}\left\{k_r\begin{bmatrix} r-1 \\ p \end{bmatrix} + k_1 k_{r-2}\begin{bmatrix} r-2 \\ p \end{bmatrix} + k_3 k_{r-4}\begin{bmatrix} r-2 \\ p \end{bmatrix} + \cdots \right.$$

$$\left. + k_{r-2}k_1\begin{bmatrix} r-2 \\ p \end{bmatrix} - \begin{bmatrix} r-2 \\ p \end{bmatrix} k_r \right\}$$

$$= \pm \frac{p+1}{p+2}\begin{bmatrix} r-2 \\ p \end{bmatrix}[k_r(p-r+1) + (k_1k_{r-2} + k_3k_{r-4} + \cdots + k_{r-2}k_1 - k_r]$$

$$= \pm \frac{p+1}{p+2}\begin{bmatrix} r-2 \\ p \end{bmatrix}[k_r(p-r) + (r+2)k_r], \text{ by (2)}$$

$$= \pm (p+1)\begin{bmatrix} r-2 \\ p \end{bmatrix} k_r.$$

the form required if (A) is to hold when p is replaced by $p+1$. Q.E.D.

REMARK: It may be worth noting that formula (A) readily gives the sum of any series of which the *nth* term is a polynomial in n. For instance, suppose $S(n)=1\cdot3\cdot4+2\cdot4\cdot5+\cdots+n(n+2)(n+3)$. Then $P(n)=n(n+2)(n+3)$ $=n^3+5n^2+6n$ and

$$S(n) = \sum_1^n P(n) = S^{(3)} + 5S^{(2)} + 6S^{(1)} = \frac{1}{12}(3n^4 + 26n^3 + 69n^2 + 46n)$$

2. Relation of the k's to the Bernoulli numbers. Writing the result in the form,

$$\sum_{m=1}^{n} m^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + p! \sum_{r=1}^{[p/2]} \frac{(-1)^{r-1}k_{2r-1}}{(p+1-2r)!} n^{p-2r+1},$$

and comparing it with the known result,

$$\sum_{m=1}^{n} m^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{1}{p+1} \sum_{r=1}^{[p/2]} (-1)^{r-1}\binom{p+1}{2r} B_{2r} n^{p-2r+1},$$

where $\binom{p+1}{2r}$ is the usual symbol for a binomial coefficient, it may be seen that

$$(2r)!k_{2r-1} = B_{2r}, \text{ or } k_s = \frac{B_{s+1}}{(s+1)!} \ (s \text{ odd}).$$

This, together with the relation

$$(2r+3)k_{2r+1} = (k_1k_{2r-1} + k_3k_{2r-3} + \cdots + k_{2r-1}k_1)$$

which is what relation (2) becomes when r is replaced by $2r+1$, gives the Bernoulli number relation

$$(2r+3)\frac{B_{2r+2}}{(2r+2)!}=\frac{B_2B_{2r}}{2!(2r)!}+\frac{B_4B_{2r-2}}{4!(2r-2)!}+\cdots+\frac{B_{2r}B_2}{(2r)!2!},$$

from which the B's can be readily calculated, starting with $B_2=2k_1=1/6$.

SOME LINES AND CURVES ASSOCIATED WITH THE PLANE TRIANGLE

By ROSCOE WOODS, University of Iowa

Consider the triangle ABC. Let a, b, c, be the sides opposite the vertices A, B, C, respectively. Take a point L in the plane of the triangle and draw through L lines parallel to the sides a, b, c. These lines determine two points on each side of the triangle. Let P, P', Q, Q', R, R' be the points determined on the sides a, b, c, respectively, so that the parallel to c cuts a in P, the parallel to a cuts b in Q, etc.; and similarly the parallel to b cuts a in P', etc. It is known that when P, Q, R are collinear[1] the point L lies on an ellipse circumscribing the triangle ABC whose centroid is the center of the ellipse. The tangent to the ellipse at each vertex of the triangle is parallel to the opposite side. This ellipse is known as the ellipse of Steiner. A great many properties of this ellipse are known and have been listed.[2] It is easy to show that when P, Q, R are collinear so are P', Q', R' and conversely.

A similarity to the theorem concerning the pedal triangle of the point L is noticed here.[3] That is, L lies on a circle when its pedal triangle has a constant area but when this area is zero this circle becomes the circumcircle. In the present case if the triangle PQR has a constant area, L lies on an ellipse whose center is the centroid of the triangle ABC, but becomes Steiner's ellipse when the area is zero. If L be placed at the centroid of the triangle ABC, it is the centroid of the triangles PQR and $P'Q'R'$. If the triangle ABC is equilateral, the ellipse degenerates into the circumcircle of the triangle. On the other hand if the ellipse of Steiner is projected into a circle the triangle ABC projects into an equilateral triangle. This observation shows that the triangle ABC is a triangle of maximum area inscribed in the ellipse. Hence if we start with the ellipse whose equation is $a^{-2}x^2+b^{-2}y^2=1$, the coordinates of the vertices

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[1] Rouché et de Comberousse, *Traité de Géométrie*, (1900 ed.) vol. 2, p. 662. Also Casey, *Analytic Geometry of the Point, Line and Circle* (1893), p. 451.

[2] Mathesis, (2), vol. 5, p. 42, p. 81; (2), vol. 7, p. 88; (4) vol. 1, p. 243.

[3] Coolidge, *A Treatise on the Circle and Sphere*, p. 49; Mathesis, (3), vol. 7, p. 99.

of the single infinity of triangles of maximum area inscribed in the ellipse are given by $a\cos(\theta+2n\pi/3)$, $b\sin(\theta+2n\pi/3)$ for $n=0, 1, 2$. The coordinates of the point L when on the ellipse are $(a\cos\phi, b\sin\phi)$. Many properties of this configuration are easily investigated since both parameters θ and ϕ occur in virtually all the equations with which we deal. If either parameter is fixed and the other varied many loci and envelopes are obtained. It is known that the lines PQR and $P'Q'R'$ envelop hypocycloids of three cusps when L generates the ellipse.[4] Their common point of intersection likewise generates a similar curve. The remainder of this paper gives the parametric equations of these curves and some of their properties and exhibits some other loci believed to be new in this connection.

The equations of the lines PQR and $P'Q'R'$ are, respectively,

$$(1)\quad (\sqrt{3})\frac{x}{a}\cos\left[\frac{3\theta-\phi+(2\pi/3)}{2}\right]+(\sqrt{3})\frac{y}{b}\sin\left[\frac{3\theta-\phi+(2\pi/3)}{2}\right]+\sin\left[\frac{3\theta-3\phi-(2\pi/3)}{2}\right]=0,$$

$$(2)\quad (\sqrt{3})\frac{x}{a}\cos\left[\frac{3\theta-\phi+(4\pi/3)}{2}\right]+(\sqrt{3})\frac{y}{b}\sin\left[\frac{3\theta-\phi-(4\pi/3)}{2}\right]+\sin\left[\frac{3\theta-3\phi+(2\pi/3)}{2}\right]=0.$$

Setting $\phi=\theta+(2n\pi/3)$ $(n=0, 1, 2)$ in (1) and (2), they furnish the sides of the triangle ABC.

These lines contain both the parameters θ and ϕ. Let θ be held constant and let ϕ vary; then the lines (1) and (2) envelop hypocycloids of 3 cusps. The parametric equations of these curves are found to be

$$(3)\quad \begin{aligned} x\sqrt{3}/a &= +\sin(3\theta-2\phi)+2\sin[\phi+(2\pi/3)],\\ y\sqrt{3}/b &= -\cos(3\theta-2\phi)-2\cos[\phi+(2\pi/3)]; \end{aligned}$$

$$(4)\quad \begin{aligned} x\sqrt{3}/a &= +\sin(3\theta-2\phi)+2\sin[\phi+(\pi/3)],\\ y\sqrt{3}/b &= +\cos(3\theta-2\phi)-2\sin[\phi+(\pi/3)]. \end{aligned}$$

That these are the equations of three-cusped hypocycloids may be shown by rotating the axes through the proper angle, for this will give rise to the standard form of the parametric equations of these curves.

[4] Mathesis, vol. 2, p. 42. M. Barbarin shows that, if the triangle ABC be projected orthogonally on a plane so chosen that it projects into an equilateral triangle, the lines PQR and $P'Q'R'$ project into generalized Simson lines. He treats this topic on p. 106 and p. 122 of this volume.

The coordinates of the cusps of these curves are

$$(5)\quad x/a = \sqrt{3}\sin[\theta + 2(3n+2)\pi/9], \quad y/b = \sqrt{3}\cos[\theta + 2(3n+2)\pi/9];$$

$$(6)\quad x/a = -\sqrt{3}\sin[\theta + 2(3n+1)\pi/9)], \; y/b = +\sqrt{3}\cos[\theta + 2(3n+1)\pi/9];$$

where $n=0, 1, 2$. It is now evident that these points all lie on the ellipse whose equation is $a^{-2}x^2+b^{-2}y^2=3$ for all θ.

These hypocycloids pass through the vertices of the triangle ABC. One of them is tangent to the sides of the triangle at the vertices, that is, tangent to c, a, b at A, B, C, respectively; while the other is tangent to b, c, a at A, B, C, respectively. Easy analysis shows that the lines (1) and (2) are never parallel but are perpendicular when $-\cos(3\theta-\phi)=(a^2+b^2)/2(a^2-b^2)$. The angle $3\theta-\phi$ will be real if $a \geqq b\sqrt{3}$, which is also the condition that at least one of the triangles ABC may be a right triangle. It is clear that the lines (1) and (2) meet in a point whose coordinates are given by

$$(7)\quad x\sqrt{3}/a = 2\cos\phi + \cos(3\theta - 2\phi), \quad y\sqrt{3}/b = 2\sin\phi + \sin(3\theta - 2\phi).$$

These are the parametric equations of a three-cusped hypocycloid whose cusps are situated at the vertices of the triangle ABC. The cusp-tangents pass through the centroid of the triangle.

Consider ϕ constant and seek the envelopes of the lines (1) and (2). It is found that these lines revolve around the points whose equations are

$$(8)\qquad \sqrt{3}x/a = \cos(\phi + \pi/6), \quad \sqrt{3}y/b = \sin(\phi + \pi/6);$$

$$(9)\qquad \sqrt{3}x/a = \cos(\phi - \pi/6), \quad \sqrt{3}y/b = \sin(\phi - \pi/6).$$

From these we see that these points always lie on the ellipse whose equation is $a^{-2}x^2+b^{-2}y^2=1/3$.

The lines associated with the extremities of a diameter of the ellipse are considered next. They are found from (1) and (2) by replacing ϕ by $\phi+\pi$. They are

$$(10)\quad (\sqrt{3})\frac{x}{a}\cos\left[\frac{3\theta - \phi - (\pi/3)}{2}\right] + (\sqrt{3})\frac{y}{b}\sin\left[\frac{3\theta - \phi - (\pi/3)}{2}\right] + \sin\frac{3}{2}[\theta - \phi + (\pi/9)] = 0,$$

$$(11)\quad (\sqrt{3})\frac{x}{a}\cos\left[\frac{3\theta - \phi + (\pi/3)}{2}\right] + (\sqrt{3})\frac{y}{b}\sin\left[\frac{3\theta - \phi + (\pi/3)}{2}\right] + \sin\frac{3}{2}[\theta - \phi + (5\pi/9] = 0.$$

The lines (1) and (10) meet in a point on the ellipse $a^{-2}x^2+b^{-2}y^2=1/3$. The lines (1) and (11) meet in the point given by

$$(12)\quad x\sqrt{3}/a=-(\sqrt{3})\cos(3\theta-2\phi)-2\sin\phi,\quad y\sqrt{3}/b=\sqrt{3}\sin(3\theta-2\phi)+2\cos\phi.$$

These are the parametric equations of a hypotrochoid. By rotation of axes this curve can be brought into a standard form and its double points easily found. It is evident that (2) and (11) meet in a point on the same ellipse as do lines (1) and (10); their point of intersection being diametrically opposite. Likewise it is clear that the lines (2) and (10) generate a second hypotrochoid.

If ϕ be replaced by $\phi+(\pi/2)$ in (1) and (2) we have lines, which we shall refer to as (13) and (14), associated with the adjacent extremities of a pair of conjugate diameters. The intersection of (1) and (13) give rise to the hypotrochoid whose equations are

$$(15)\quad \begin{aligned} x\sqrt{3}/a &= -\cos(3\theta-2\phi)+(\sqrt{2})\cos[\phi+(5\pi/12)] \\ y\sqrt{3}/b &= -\sin(3\theta-2\phi)+(\sqrt{2})\sin[\phi+(5\pi/12)]. \end{aligned}$$

Again lines (1) and (14) meet on the same ellipse as do (1) and (10). Similar results are obtained when (2) and (13) and (14) are paired.

The sum of the eccentric angles of four concyclic points on an ellipse is a a multiple[5] of 2π. The fourth point D on an ellipse determined by the circumcircle of an inscribed triangle ABC of maximum area is known as Steiner's point.[6] The eccentric angle D is therefore $-3\theta+2n\pi$. A close resemblance between the equation of line (1) and the equation of the chord LD is readily seen. Hence considering these two lines together we find their common intersection to lie on a conic when θ varies and on a quartic curve when ϕ varies. This quartic has 3 or 1 real branches through D according as D lies within or without the hypocycloid (3). This quartic has branches that extend to the infinite region of the plane. Similar remarks apply to LD and (2). Consider the line $L'D'$ symmetrical to LD with respect to the x-axis. For variable ϕ the common intersection of (1) and $L'D'$ is a quartic curve lying in the finite part of the plane with 3 or 1 real branches through D' according as D' lies within or without the hypocycloid (3). For variable θ this intersection is a conic. For the line passing through the origin always parallel to $L'D'$, the quartic generated is similar to the three-leaved rose and becomes a three-leaved rose when the triangle is made equilateral. The conics in this case are ellipses with their centers on the ellipse $a^{-2}x^2+b^{-2}y^2=1/9$. There are evidently other interesting loci when other lines in the configuration are taken.

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[5] C. Smith, *Conic Sections by the Methods of Coordinate Geometry* (1914), p. 169.

[6] Steiner, Gesammelte Werke, vol. 2, p. 691.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. Buchanan, Tulane Univerity, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. REMARKS ON ORTHOLOGIC TRIANGLES

By N. AGRONOMOF, University of Vladivostok

1. It is known that

(1) if a_1, a_2, a_3 are the feet of perpendiculars from a point M on the sides A_2A_3, A_3A_1, A_1A_2 of a triangle $A_1A_2A_3$, then[1]

$$(A_2a_1)^2 + (A_3a_2)^2 + (A_1a_3)^2 = (A_3a_1)^2 + (A_1a_2)^2 + (A_2a_3)^2$$

and (2) if S is any point in the plane of the triangle $A_1A_2A_3$ and if any straight lines B_2B_3, B_3B_1, B_1B_2, be drawn perpendicular respectively to SA_1, SA_2, SA_3, so that the perpendiculars from A_1, A_2, A_3 on the sides of $B_1B_2B_3$ are concurrent (at S), then shall the perpendiculars from $B_1B_2B_3$ on the sides $A_1A_2A_3$ be concurrent.[2]

Triangles $A_1A_2A_3$, $B_1B_2B_3$ which are thus related are said to be mutually orthologic.

2. Now we consider two triangles $A_1A_2A_3$ and $B_1B_2B_3$ in a plane. Let a_1, a_2, a_3 be the feet of the perpendiculars from B_1, B_2, B_3 on the sides A_2A_3, A_3A_1, A_1A_2; and let b_1, b_2, b_3 be the feet of the perpendiculars from A_1, A_2, A_3 on the sides B_2B_3, B_3B_1, B_1B_2. We call the *steiner* of $A_1A_2A_3$ with respect to $B_1B_2B_3$ the expression

$$S_{ab} = (A_2a_1)^2 + (A_3a_2)^2 + (A_1a_3)^2 - (A_3a_1)^2 - (A_1a_2)^2 - (A_2a_3)^2.$$

Similarly

$$S_{ba} = (B_2b_1)^2 + (B_3b_2)^2 + (B_1b_3)^2 - (B_3b_1)^2 - (B_1b_2)^2 - (B_2b_3)^2$$

is the *steiner* of $B_1B_2B_3$ with respect to the triangle $A_1A_2A_3$.

We next prove this theorem: *The steiner of $A_1A_2A_3$ is the negative of the steiner of $B_1B_2B_3$.* Let x_1, y_1; x_2, y_2; x_3, y_3 be the coordinates of the points A_1, A_2, A_3 and let x_1', y_1'; x_2', y_2'; x_3', y_3' be the coordinates of the points B_1, B_2, B_3. It follows that the *steiner* S_{ab} is

$$2\alpha_1(x_3 - x_2) + 2\alpha_2(x_1 - x_3) + 2\alpha_3(x_2 - x_1) + 2\beta_1(y_3 - y_2) + 2\beta_2(y_1 - y_3) + 2\beta_3(y_2 - y_1),$$

[1] T. Steiner, Annales de Mathématiques de Gergonne, vol. 19 (1828–29), p. 37.

[2] W. Gallatly *The Modern Geometry of the Triangle*, pp. 55–56.

where $\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3$ are the coordinates of the points a_1, a_2, a_3. The equations of B_1a_1, B_2a_2, B_3a_3 are

$$x(x_3 - x_2) + y(y_3 - y_2) = x_1'(x_3 - x_2) + y_1'(y_3 - y_2),$$

. . . . . .

. . . . . .

whence

$$\alpha_1(x_3 - x_2) + \beta_1(y_3 - y_2) = x_1'(x_3 - y_2) + y_1'(y_3 - y_2)$$

. . . . . .

. . . . . .

The above set of equations then becomes

$$\begin{aligned} S_{ab} &= 2[\sum x_1'(x_3 - x_2) + \sum y_1'(y_3 - y_2)] \\ &= 2[\sum x_1(x_2' - x_3') + \sum y_1(y_2' - y_3')] = -S_{ba}. \end{aligned}$$

This proves the theorem.

3. Putting $S_{ab}=0$, we obtain the theorem (2) of §1.

It is seen from the preceding that the theorem offered by me belongs to the same category as the theorems which Mr. Paul Wernicke has given in his article on *The theorems of Ceva and Menelaus and their extension.*[1]

II. The Summation of a Class of Series

By Joseph D. Grant, University of Illinois

Many, doubtless, have noticed that the series for $\log_e 2$ is made up of alternate terms from the series:

$$1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots.$$

Here it is proposed to show a simple means for summing certain series made up of terms from the series:[2]

$$(1) \qquad \sum_{x=0}^{\infty} \frac{1}{(ax+d)^{\lfloor n \rfloor a}} = \frac{1}{a(n-1)(ax+d)^{\lfloor n-1 \rfloor a}}\Bigg]_{x=0}, \quad a, \quad d \geqq 1, \quad n \geqq 2,$$

wherein $(ax+d)^{\lfloor n \rfloor a} = (ax+d)(ax+a+d)(ax+2a+d)\cdots(ax+na-a+d)$. Grouping the terms of (1) in sets of c terms each, one has

$$(2) \qquad \begin{aligned} \sum_{x=0}^{\infty} \frac{1}{(ax+d)^{\lfloor n \rfloor a}} &= \sum_{x=0}^{\infty} \sum_{k=0}^{c-1} \frac{1}{(acx+ak+d)^{\lfloor n \rfloor a}} \\ &= \sum_{k=0}^{c-1} \sum_{x=0}^{\infty} \frac{1}{(acx+ak+d)^{\lfloor n \rfloor a}}. \end{aligned}$$

[1] This Monthly, vol. 34 (1927), pp. 468–472.

[2] Chrystal's Algebra, (1920 edition), vol. 2, p. 403.

The series (1) is thus separated into c series which may be distinguished by four variable parameters, a, $ak+d$, c, n. Hence it is convenient to define

$$f(a,b,c,n) = \sum_{x=0}^{\infty} \frac{1}{(acx+b)^{|n|a}}, \text{ where } b = ak + d. \tag{3}$$

The purpose of this paper is to prove that

$$a^{n-1}(n-1)!\, f(a,b,c,n) = \int_0^1 v^{b-1} \frac{(1-v^a)^{n-1}}{(1-v^{ac})}\, dv, \quad n \geqq 2. \tag{4}$$

Restating (2) in terms of $f(a, b, c, n)$, we find

$$f(a,d,1,n) = \sum_{k=0}^{c-1} f(a,ak+d,c,n). \tag{5}$$

If one writes out and examines the general terms of the series it may be seen that

$$f(a,b,c,n) - f(a,a+b,c,n) = a \cdot n \cdot f(a,b,c,n+1). \tag{6}$$

Before proceeding to prove (4), it should be noted that the value there given for $f(a, b, c, n)$ is consistent with (5) and (6).

Consider the identity:

$$\begin{aligned} v^{b-1}(1-v^a)/(1-v^{ac}) \equiv{}& v^{b-1} - v^{a+b-1} + v^{ac+b-1} - v^{ac+a+b-1} + \cdots \\ &+ v^{acw+b-1} - v^{acw+a+b-1} + \frac{v^{acw+b-1}\cdot(1-v^a)}{(1-v^{ac})}. \end{aligned} \tag{7}$$

Multiplying (7) through by dv and integrating throughout between the limits 0 and 1, we find

$$\begin{aligned} \int_0^1 \frac{v^{b-1}(1-v^a)}{(1-v^{ac})}\, dv = {}& \frac{1}{b} - \frac{1}{a+b} + \frac{1}{ac+b} - \frac{1}{ac+a+b} + \cdots \\ &+ \frac{1}{acw+b} - \frac{1}{acw+a+b} + \int_0^1 \frac{v^{acw+b-1}(1-v^a)}{(1-v^{ac})} dv. \end{aligned} \tag{8}$$

Combining the terms of (8) in pairs and referring to the second integral as R, we obtain

$$\int_0^1 \frac{v^{b-1}(1-v^a)}{(1-v^{ac})}\, dv - R = a \cdot \sum_{x=0}^{w} \frac{1}{(acx-b)^{|2|a}}. \tag{9}$$

From (3) it is seen that

$$\lim_{w\to\infty} \sum_{x=0}^{w} \frac{1}{(acx+b)^{|2|a}} = f(a,b,c,2).$$

R is a function of w and may be written:

$$R = \int_0^1 \frac{v^{aw+\beta}(1 - v^a)}{(1 - v^{ac})}\, dv.$$

For v in the interval (0, 1), $(1-v^a)/(1-v^{ac}) \leqq 1$, since $a \geqq 1, c \geqq 1$. Hence $R \leqq \int_0^1 v^{aw+\beta} dv = (\alpha w+\beta+1)^{-1}$.

Since R is always less than, or at the most equal to, a quantity which approaches zero as w approaches infinity, it follows that $\text{limit}_{w\to\infty} R=0$, and (4) is established for $n=2$.

Equation (4) may now be established, in general, by mathematical induction, using (6) to pass from n to $n+1$, and the value established for $n=2$ as the special case.

The formulas (3), (4), (6) above may be generalized to the following, the proof being the same as before.

$$(3') \qquad f(a,b,c,n,\rho) = \sum_{x=0}^{\infty} \frac{1}{(acx + b)^{|n|a}} \left(\frac{1}{\rho}\right)^{acx+b}, \qquad \rho \geqq 1,$$

$$(4') \qquad a^{n-1}(n - 1)!\, f(a,b,c,n,\rho) = \int_0^{1/\rho} \frac{x^{b-1}(1 - \rho^a x^a)^{n-1} dx}{(1 - x)^{ac}},$$

$$(6') \qquad f(a,b,c,n,\rho) - \rho^a f(a,a + b,c,n,\rho) = anf(a,b,c,n + 1,\rho).$$

Many interesting results may be derived from (4). One of the simplest follows from the value it gives for the sum in (1):

$$\int_0^1 v^{b-1}(1 - v^a)^{n-2} dv = \sum_{R=0}^{n-2} \frac{{}_{n-2}C_R}{aR + b}(-1)^R = \frac{a^{n-2}(n - 2)!}{(ax + b)^{|n-1|a}}\Big]_{x=0}.$$

In conclusion: It has been shown that an interesting class of series may be summed by integration such as is treated in every elementary text in the calculus.

III. Two Notes on Cyclic Cubics

By Raymond Garver, University of California at Los Angeles

1. A General Expression for Cyclic Cubics. Seidelmann[1] has given general expressions which represent all cubic and quartic equations with rational coefficients which have certain Galois groups for the field of rational numbers. His article is simply a summary of results given in his Dissertation, and does not give detailed methods; his general method, however, as outlined near the beginning of his paper, involves the use and manipulation of functions "belonging" to the required group.

[1] Mathematische Annalen, vol. 78 (1918), pp. 230–233.

I wish to show that his expression for the general cyclic cubic with second term removed,

$$x^3 - 3(p^2 + 3q^2)x + 2p(p^2 + 3q^2) = 0, \tag{1}$$

can be obtained in a very elementary manner. In fact if we define a cyclic cubic as an irreducible cubic whose discriminant is a perfect square, we may avoid all reference to groups.

First, it is clear that any cubic with rational coefficients whose second term has been removed by a linear transformation is of the form

$$x^3 + Am^2x + Am^3 = 0,$$

where A and m are rational. If this equation is cyclic, its discriminant is a perfect square, which is equivalent to saying that we must have

$$A = -(c^2 + 27)/4, \text{ where } c \text{ is rational.}$$

If we now put $p = -3m/2$, $q = cm/6$,[1] the equation takes Seidelmann's form. Equation (1) hence represents all cyclic cubics, first, since every cyclic cubic has been shown to be of that form and, secondly, since every cubic of form (1) is cyclic (if it is irreducible), having its discriminant a perfect square.

2. A Normal Form for Cyclic Cubics. There are a number of simple one-parameter normal forms to which a cyclic cubic can be reduced. Cajori[2] gives the form

$$z^3 - 3(m^2 + m + 1)z + (m^2 + m + 1)(2m + 1) = 0,$$

which becomes, on putting $2m+1 = P$, $y = 2z$,

$$y^3 - 3(P^2 + 3)y + 2P(P^2 + 3) = 0. \tag{2}$$

This form resembles (1) closely, but is a normal form and not a general expression.

Cajori's derivation of his normal form is long, and gives no method by which the reduction can be carried out for a *particular* cubic. These two objections seem to be met by the following theorem:

Any cyclic cubic can be transformed into form (2) by a quadratic Tschirnhaus transformation with rational coefficients.

To show this, it is necessary merely to consider the general cyclic cubic in the form (1), and to apply the transformation

$$y = -p^{-1}q^{-1}[x^2 + px - 2(p^2 + 3q^2)].$$

The coefficients of the transformed equation are obtained with the help of formulas which I gave recently in this Monthly:[3] I shall not reproduce the calculations here. Equation (2) is easily obtained upon putting $P = -3q/p$.

[1] i.e., $m = -2p/3$, $c = -9q/p$.

[2] *Theory of Equations*, 1921, pp. 196–197.

[3] Vol. 34 (1927), pp. 521–522.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB TOPICS

1928 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By WALTER CROSBY EELLS, Stanford University, California.

In continuation of previously published lists of centennial dates[1] in the history of mathematics, the following list of important 1928 centennial dates is presented. They may be of interest and value to mathematics teachers as suggestions for program topics or special investigations and reports. Further details may be found in the standard histories of mathematics.

628. Publication of the *Brahma-sphuta-siddhanta* of Brahma-gupta, most prominent Hindu mathematician of the seventh century, with whom mathematics in India reached its high water mark.

1228. Publication of the revised edition of Leonardo of Pisa's *Liber Abaci* which marked the first renaissance of mathematics on Christian soil, introduced Arabian algebra, and brought into general use in Europe the labor-saving Hindu-Arabic numerals, and for centuries was a storehouse of material for later European writers on arithmetic and algebra.

1528. Death of Albrecht Dürer, celebrated artist, author of first printed work on higher plane curves, and earliest occidental writer to call attention to magic squares. See the magic square in his well known painting "Melancholia."

1528. Death of Johannes Werner, author of the first book on conic sections in Christian Europe.

1628. Publication of Adrian Vlacq's fourteen place logarithm tables of numbers from 1 to 100,000, of which 70,000 were computed by him.

1728. Birth of J. H. Lambert, versatile mathematician who first proved the irrationality of pi.

1828. The symbol "e" for the natural base of logarithms introduced by Euler.

[1] This Monthly, vol. 33 (1927), p. 141 for a list of 1927 events and for references to previous issues for corresponding lists for 1926 and 1925.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED*

Plane Trigonometry, by ARTHUR M. HARDING and GEORGE W. MULLINS. New York, the Macmillan Company. 118 pages.

A concise and conservative text.

College Algebra, by CLAUDE IRWIN PALMER and WILSON LEE MISER. New York, McGraw-Hill Book Company. xiv+377 pages. $2.50.

Another conservative book, unusually full in treatment and sufficiently elastic for a brief or an extended course.

Analytic Geometry, by R. L. BORGER. New York, McGraw-Gill Book Company. xii+334 pages.

Interesting features of this text are the emphasis on ruler and compass construction, the simultaneous study of cartesian, polar and parametric equations, and the introduction of the derivative.

Short Course in Spherical Trigonometry, by PAULINE SPERRY. Richmond, Johnson Publishing Company. xii+60 pages.

"An effort to meet the . . . . need for a short yet rigorous text."

Mathematics in Liberal Education, by FLORIAN CAJORI. Boston, Christopher Publishing House. 169 pages. $1.50.

"A critical examination of the judgments of prominent men of the ages" as to the values of mathematics.

The Symmetrical Optical System, by G. C. STEWARD. Cambridge Tracts in Mathematics and Mathematical Physics, No. 25. Cambridge University Press. 102 pages.

Beyond the Electron, by SIR J. J. THOMSON. Cambridge University Press. 43 pages.

A lecture on the new physics, delivered at Girton College.

Theory of Probability, by WILLIAM BURNSIDE. Cambridge University Press. xxx+106 pages.

"A manuscript which Professor Burnside had practically completed some time before his death. It has been edited for the press, with a memoir, by Dr. A. R. Forsyth."

The New Quantum Mechanics, by G. BIRTWISTLE. Cambridge University Press. xiv+290 pages.

"This book is concerned with the development of quantum mechanics during the past two years."

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* The quotations given below, unless otherwise specified, are from prefaces and introductions to the books named.

Neuere Methoden und Ergebnisse in der Hydrodynamik, VON C. W. OSEEN (Upsala). Leipzig, Akademische Verlagsgesellschaft M.B.H. xxiv+337 pages.

This is Band I of a series designated as "Mathematik und Ihre Anwendungen in Monographen und Lehrbüchern," edited by E. Hilb of the University of Würzburg. xxiv+337 pages.

Plane and Spherical Trigonometry, by MURRAY J. LEVENTHAL. New York, Globe Book Company. $0.53.

The purpose of this book is to present the essentials of trigonometry in as brief a form as possible.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Researches into the Mathematical Principles of the Theory of Wealth. By AUGUSTIN COURNOT. Translated by Nathaniel T. Bacon, with an essay on "Cournot and Mathematical Economics" and a bibliography of mathematical economics by Irving Fisher. The Macmillan Co., 1927. xxiv+213 pages.

This translation of Cournot's classic of 1838 was first published in 1897. The present volume is an exact reprint, except for the addition by Professor Fisher of a foreword and twelve pages of notes on Cournot's mathematics. With his usual clearness Professor Fisher helps the reader of moderate mathematical attainments over the more difficult steps. In the foreword he says:

> In the twenty years since the book first appeared in English, the mathematical method has become so general in economic and statistical studies that no attempt has been made to bring the bibliography down to date by adding the many items which would be necessary, and there is today little need, as there then was, to emphasize the value of that method, as it is now seldom, if ever, challenged.

This is not the place to go into the criticisms and prolonged discussion of Cournot's theories of duopoly and of foreign exchange. If students continue to believe in the literal inspiration of textbooks, the book should not be used as a text. But the work has a value not merely historical, for it sets an example of precise reasoning in economics which is worthy of emulation.

The publication of this edition is related to a widespread increase in interest in mathematical economics; and by mathematical economics is meant not only statistics, but the development with free use of mathematical tools of the consequences of economic assumptions. The work of Cournot, Jevons, Walras, Pareto, and Marshall has stimulated two main lines of development. The more widely known is the study of economic statistics and the investigation of particular industries, with the object more or less plainly in view of obtaining definite expressions for supply and demand functions. This work has become allied with the financially attractive enterprise of business forecasting, but has encountered practical difficulties; only a few of the functions in question have thus far been evaluated, and these with dubious reliability. On the other hand is the theoretical work represented by Irving Fisher's doctoral thesis, recently

republished, and by his studies of interest rates, currency and the price level; by the works of Edgeworth, Pantaleoni, and Amoroso; and now by C. F. Roos, who has actually uncovered some *mathematically* new problems. Contributions of the deductive type have been made by G. C. Evans and others. Like mathematical physics and laboratory experiment, the two pursuits must henceforth go on side by side, each deriving meaning and guidance from the other.

Incidentally this trend in economics is demanding more and earlier mathematics. The prospective economist who is not familiar with calculus before his sophomore year is now at a disadvantage. Both in economic theory and in statistics the rate of learning could be quadrupled if all the students in the class could handle calculus, if texts were available assuming this condition and as well written as the average book on college algebra, and if instructors were themselves able to take advantage of the situation. The economist has small use for conic sections or for drill in solving triangles, but he does need calculus. The same is true of the sociologist, the political scientist, the psychologist, and the biologist. If alongside the familiar curriculum we could have one in which trigonometry should be regarded primarily as preparation for calculus rather than for surveying, and in which integral calculus were completed in the first college year or earlier, a serious need of these important groups could more easily be met. The graduate student who now so commonly turns aside from the subject of his choice to struggle with elementary mathematics is probably acting wisely, but the diversion of attention from his main interest must make the results less valuable than if his mathematics were already well enough consolidated to allow him to go straight ahead.

HAROLD HOTELLING

Die trigonometrischen Lehren des persischen Astronomen Abu'l-Raihân Muh. ibn Ahmad Al-Bîrûnî, dargestellt nach Al-Qânûn Al-Mas'ûdî von CARL SCHOY. Hannover, Orient-Buchhandlung Heinz Lafaire, 1927. xii+108 pages.

With the death of Carl Schoy on December 6, 1925, scholarship in the field of history of exact science suffered a severe and irreparable loss. In his short life time, Schoy had done much very valuable work in uncovering the buried treasures of Arabian mathematics, including in particular many on astronomical and geographical subjects. In his earlier years handicapped by poverty, and later by an insidious and incurable disease, Schoy's career compels our admiration for his never-failing courage and indomitable will, no less than for his scientific achievements.[1]

The book under review was finished and prepared for the press by the author in 1924, but was not published until after his death, when it was seen through

[1] Biographical sketch, by J. Ruska, *Isis*, vol. 9 (1927), pp. 83–89 with portrait.

the press by J. Ruska and H. Wieleitner. Although it is the most extensive published work of the author, he had designed it to be only a small section in a history of Arabian mathematics which he was planning to write.

Al-Biruni, whose trigonometric work is here published for the first time in any detail, was a Persian astronomer, mathematician, and geographer of the 11th century. He resided at Gazna at the time when it was the brilliant capital of Mahmud the conqueror of India. Al-Biruni traveled widely in India, and his description of this country is one of the greatest works in all Arabian literature. As a scientist he was both original and versatile, and marks, with Ibn Sina (Avicenna) the high-water mark of Moslem science. Thus the trigonometric work of Al-Biruni is of great interest and importance in the history of mathematics.

Schoy has here given us 10 chapters, selected from Al-Biruni's *Al-Qânûn al-Mas'ûdî*. Chapter 1 gives the lengths of the sides of the regular polygons of 3, 4, 5, 6, 8, and 10 sides inscribed in a circle of radius unity. Chapter 2 shows how to find the chord of double an arc from the chord of the simple arc; similarly, the chord of half or a fourth of a given arc, and in general $(1/2)^n$ of a given arc; and the chord of the sum or difference of two arcs whose chords are given. Chapter 3 gives a construction of the regular nonagon, with a computation which gives the numerical value of the length of a side (in sexagesimals) correct to 7 decimal places. At the end of the chapter, as a verification of his geometrical reasoning, Al-Biruni shows how by trigonometric combinations starting with 45° and 30° we can reach the chord of $40°0'0''0'''24^{IV}+$, which gives a value agreeing with that previously obtained, within less than one two-millionth. Chapter 4 gives two methods of finding the chord of 1° when the radius is unity: (1) by starting with s_9 and s_{10}, using $40°-36°=4°$ and halving twice; (2) by using s_{30} and s_{36}, $12°-10°=2°$, and halving. Next, the trisection equation for 3° is obtained in geometric form, and thus the chord of 1° is computed, agreeing to $(1/60)^4$ with the other results.

Chapter 5 gives a computation of π, resulting in the value 3.14174. Schoy remarks that he himself obtained the considerably more accurate value 3.14167 by using Al-Biruni's methods with but a very slight modification. Chapter 6 explains the construction of Al-Biruni's trigonometric table of sines, which is also printed therewith (pp. 37–39.) The angles progress by quarter-degrees; the sines are given to fourths (in sexagesimal notation) thus: $\sin 2°30' = 2'37''1'''47^{IV} = .043618+$. Chapter 7 explains the use of the table in finding the sine or versed sine of a given angle, or the angle corresponding to a given sine or versed sine. Second differences are taken into account in the process. Chapter 8 deals with the "shadows of illuminated objects" and explains the solution of problems involving tangents, after the manner of Abu'l Wafa. A table of tangents at intervals of 1° with first and second differences follows. At

the end of the chapter the law of sines is proved. Chapters 9 and 10 contain the proof of the sine law for a spherical triangle, and a few other theorems derived from it.

The remainder of the book (pp. 64–108) consists of supplementary material, included by the author because of its close relation to Al-Biruni's work. First, Al-Biruni's own application of his theorems on spherical trigonometry to problems of practical astronomy; secondly, Tabit ibn Qurra's and Ibn Al-Haitam's monographs on the regular heptagon; and thirdly, the sine and tangent tables of Ulug Beg.

This book is a very valuable addition to the primary sources of information about Arabian mathematics, and emphasizes the greatness of the loss which its author's premature death has caused.

R. B. McClenon

Calculus. By H. B. Fine. The Macmillan Co., 1927. 418 pages.

The Calculus. By R. D. Carmichael and J. H. Weaver. Ginn and Co. 342 pages.

Differential and Integral Calculus. By A. Cohen. D. C. Heath Co. 540 pages.

These three books cover the usual topics in calculus in about the customary manner.

Fine's book has in addition some material usually reserved for texts in advanced calculus, including chapters on Fourier series and functions of a complex variable. The distinctive feature is the care with which questions of rigor are treated. For example, the author says that the slope of the tangent to a curve at a point is by *definition* the limit, if it exists, of the slope of the secant, and that the area under a curve is by *definition* the limit of a certain sum. It might seem superfluous to mention such things in 1928, but for the fact that the other two books are written as if slopes of curves and areas already existed, previous to definition, and as if mathematicians had been able to find their values by ingenuity and insight. Of course everyone wants books for beginners to appeal a great deal to the intuition and to avoid proofs and distinctions which the students cannot understand. The point is that Fine has avoided saying things which will have to be unlearned and has at the same time kept the appeal to the intuition. He does this by the turn of a phrase, without formality. The formal proofs occupy surprisingly few pages in all, and yet everything is there. The book is condensed, partly by leaving to the teacher some of the pleasure of making illuminating remarks and telling what it is all about, partly by a skillful arrangement of material, and partly by including fewer worked out examples. The exposition does not seem harder to read than other much longer expositions which skip half the reasoning and get the rest mixed up.

The text of Carmichael and Weaver is written in an enthusiastic and vivid style. There is a conscious, and in the reviewer's opinion, successful effort to keep the student's work from degenerating into formal manipulation. For example, more space than usual is given to making the reader associate some geometrical or kinetic picture with the word derivative and the symbol dy/dx, before formal differentiation is taken up. Perhaps as a result of this there are a good many phrases, especially in the opening chapter which will offend the purists. The following is typical, (p. 12)—"By the slope of a curve at a given point we mean the slope of the tangent line to the curve at that point." Although this comes in a section which is frankly untechnical, there are two objections to it. First, the slope of a curve is as vivid intuitionally as the existence of a tangent; and secondly, it is apparently and not actually a definition. It might just as well read—"By the tangent line to a curve at a point we mean a straight line through the point having the same slope as the slope of the curve at that point." Neither sentence adds to the clearness or to the reasoning. On the other hand, the author's habit of stopping occasionally to tell where he is going makes the book easier reading and more interesting.

While Fine confines himself to saying things completely and just once, Cohen, like Carmichael and Weaver, writes as a teacher talks in the classroom, giving motives and purposes, sometimes mentioning mistakes to be avoided, repeating and paraphrasing. "Much emphasis is laid upon applications to geometry, mechanics, and the physical sciences." The exposition is full and both the verbal and algebraic parts are given in enough detail for moderately good students. This text also may be criticized on logical grounds. For one of many possible examples, on page 229, there is a page of algebra ending with the formula for the differential of arc length. Anyone mature enough to read this would also be mature enough to realize that the length of arc requires definition, and to ask if the length of an arc "always concave to its chord" is necessarily less than the length of a broken line on the convex side with the same end points. It would be pedagogically better and logically the same to write at once, $ds^2 = dx^2 + dy^2$. There are numerous misprints. There are plenty of worked out examples, and those left for the student include many not in other texts.

K. W. LAMSON

Analytic Geometry. By E. S. CRAWLEY and H. B. EVANS. F. S. Crofts & Co., 1928. xiii+244 pages. Price $2.00.

The present edition is a revision of the original text written in 1917.

The first seven pages are taken up with a series of formulae and tables.

The first section contains a discussion of oblique coordinates, which is perfectly logical, but is not the usual method of starting the beginner in analy-

tic geometry. No exercises as vitalizing material are given under this section. This plan of giving articles of a theoretical nature without concrete exercises immediately following each article is quite common in the text. The plan of deriving the general formula and then having the student solve exercises by applying the formula is an important way of teaching the subject; but the scheme of having a student go through, in solving a problem, the same process as the author carries out in deriving the formula is also an excellent plan from a pedagogical point of view.

The way of dividing exercises at the end of each chapter into two classes is highly commendable. The first class contains simple and direct applications of the theory given in preceding articles. The second class includes general exercises requiring more maturity on the part of the student. This plan will aid the teacher in the assignment of lessons.

In the chapter on graphs of equations the idea of showing one graph with isolated points corresponding to the pairs of values of x and y, and then another graph with a curve drawn through the points, is highly commendable as a pedagogical scheme. The plan of showing points equally spaced to represent measurements parallel to the x and y axes will be found of aid to the student working at a black board which is not cross-sectioned.

In the chapter on the straight line, the writer of this review feels that there are too many sections on the general derivation of formulae before any exercises are given. The equations of the following forms are given before any exercises are set for the student: two-point, point-slope, slope-intercept, intercept, and normal. Instead of using the normal form of the equation of a straight line to determine the perpendicular distance of a given point from a given line, the authors have a rather unique way of carrying out the derivation of the well known formula

$$d = (Ax_1 + By_1 + C)/\pm (A^2 + B^2)^{1/2}.$$

The scheme consists in writing the equation of the perpendicular from the point (x_1, y_1) to the line

$$Ax + By + C = 0,$$

and then finding the coordinates of the point of intersection of the two lines. Then the distance formula

$$d = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/}.$$

is used. In theory this plan is much simpler than the one usually followed, but the algebraic work is more difficult for the student to follow.[1]

[1] EDITOR'S NOTE: If in the class room we ask our students to suggest methods of attacking this problem, the most frequent suggestion is that of the method discussed here. Perhaps the fact that the process is one that appeals to the student mind justifies its use in spite of the rather formidable algebraic reductions.

Much more extensive treatment of imaginaries is carried out in this text than is usual. However, this should find favor with teachers, since it is highly probable that too little attention to these numbers has been given in current texts.

The introduction to the chapter on conic sections by way of geometric constructions based upon the arc tan e is very interesting. The authors have here a simple operation which holds alike for all the conics.

The chapter on tangents, normals, diameters, poles, and polars contains 20 pages, which is ample space for covering these topics in a thorough manner. Another chapter of about the same length covers the general equation of the second degree.

The discussion of polar coordinates is relegated to a later chapter in the text. The reviewer feels that it is better to introduce this subject early in the course, so that constant application may be made during the term.

The last two chapters of the book cover empirical equations and space geometry.

MYRON O. TRIPP

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3341. Proposed by Paul Wernicke, Washington, D. C.

Given a triangle ABC and a point P, find the conditions for the construction of a triangle having its sides parallel and proportional to the joins PA, PB and PC.

3342. Proposed by R. Goormaghtigh, La Louvière, Belgium.

Let α, β, γ, δ be the points where the straight lines AP, BP, CP, DP, meet the faces of the tetrahedron $ABCD$; the perpendiculars dropped from the vertices A, B, C, D on the lines joining respectively α, β, γ, δ to the orthocentre H meet the corresponding faces of the tetrahedron $ABCD$ in four coplanar points. The plane passing through these four points is perpendicular to PH.

NOTE BY THE EDITORS: This problem is a generalization of Problem **3228** [**1926, 525** and **1928, 42**]. See also the solution of **3258** [**1928, 210**].

3343. Proposed by J. V. Uspensky.

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{2n\pi}^{\infty} \frac{\sin z}{z} dz = \pi - \frac{\pi}{2} \log 2\pi$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{n\pi}^{\infty} \frac{\sin z}{z} dz = \frac{\pi}{2} - \frac{\pi}{2} \log \pi.$$

3344. Proposed by B. F. Yanney, Wooster, Ohio.

Two altitudes of a given triangle and the side from whose extremities these altitudes are drawn meet in collinear points the corresponding sides of the orthic triangle of the given triangle.

3345. Proposed by B. F. Finkel, Drury College.

A mill wheel of radius a revolves so that its rim has a velocity v, and drops of water are thrown off from the rim. Find the envelope of the paths of the drops.

3346. Proposed by Frank Irwin, University of California.

Show that, if $P_1P_2 \cdots P_n P_1$ be any polygon of n sides, the broken line whose segments are parallel and equal to $P_1 P_k$, $P_2 P_{k+1}$, $\cdots$, P_nP_{k-1} $(P_{n+i}=P_i)$ in order will close. Generalize.

Again let S_i be the middle point of P_iP_{i+1}, $i=1, 2, \cdots n$, $(P_{n+1}=P_1)$; show that the broken line whose sides are parallel and equal to P_1S_k, P_2S_{k+1}, $\cdots$, $P_n S_{k-1}$ in order will close. Generalize.

3347. Proposed by R. S. Underwood, Texas Technological College.

If $x^n+y^n=z^n$, where n is an odd positive integer greater than or equal to 3, and x, y, z are positive integers, prove that $x>6K$ and $y>6K$, where K is the product of all distinct factors in n other than 2 and 3. As a corollary, it follows immediately that, if

$$x^p + y^p = z^p \quad (p \text{ a prime} \geqq 3) \quad x > 6p \text{ and } y > 6p.$$

3348. Proposed by A. C. Aitken, University of Edinburgh.

Show that

$$\frac{\pi}{6} = \sum_{1}^{\infty} \operatorname{arccot} 2u_r^2,$$

where $u_r=4u_{r-1}-u_{r-2}$ with $u_1=1$, $u_2=3$.

SOLUTIONS

3280. [1927, 438]. Proposed by Philip Fitch, Denver Public Schools.

If a flexible chain, suspended at the ends from points in a horizontal line, is loaded so that the load varies directly as the square of the distance along the horizontal from the mid-point of the chain, find the equation of the curve made by the chain.

Solution by J. B. Reynolds, Lehigh University.

Assuming a y-axis vertical through the mid-point of the chain and an x-axis horizontal, let T be the tension in the chain at a point P, (x, y) where the tangent to the curve makes an angle ϕ with the x-axis. Then, if the load at the point P is kx^2 we have,

$$(1) \qquad T d\phi/ds = kx^2 \cos\phi, \qquad (2) \quad dT/ds = kx^2 \sin\phi.$$

Dividing these equations and integrating the result we find, $T=T_0 \sec\phi$, in which T_0 is the tension at the mid-point where $\phi=0$.

Eliminating T from (1) we get

$$T_0 \sec^2\phi \, d\phi = kx^2 ds,$$

or since $\tan\phi=dy/dx=p$,

$$T_0 dp/(1+p^2)^{1/2} = kx^2 dx.$$

Upon integrating this equation we find,

$$p = \tfrac{1}{2}[\exp kx^3/3T_0 - \exp(-kx^3/3T_0)]$$

and, therefore, for the required curve,

$$y = \int \sinh (kx^3/3T_0)dx$$

Also solved by PAUL WERNICKE.

3282 [1927, 438]. Proposed by H. Halperin, A. & M. College of Texas, College Station, Texas.

Justify the following method of solving an exact differential equation of the first order, $Mdx+Ndy=0$: Replace y in M and N by ux and dy by udx; integrate the resulting expression with regard to x, considering u as constant; replace u by y/x, equate to a constant, C, and obtain the solution.

SOLUTION BY OTTO DUNKEL, Washington University.

In the theorem of the problem the statement should be made more precise in the form "integrate the resulting expression with regard to x between the limits x and 0, considering u as a constant." In a convex region containing the origin in which M and N are continuous functions of x and y, the integral of $Mdx+Ndy=df(x, y)$ along a chosen path of a suitable form within the region from (0, 0) to the point (x, y) which is fixed for the moment is $f(x, y)-f(0, 0)$. The value of the integral does not depend upon the path, and we may use as a path the straight line $y=ux$, where u is a constant so chosen that the line goes through the other end point. The integral becomes

$$\int_0^x [M(x,ux) + uN(x,ux)]dx = \phi(x,u) - \phi(0,u).$$

Hence, replacing in the above u by y/x, we have

$$f(x,y) = \phi(x,y/x) - \phi(0,y/x) + f(0,0),$$

which is obtained by equating the results of the two integrations. This is an identity in x and y, and it proves the theorem as amended.

If the origin is a singular point as in $-(y+3)dx/x^2+dy/x=0$, the method gives an incorrect result, But we may first make the transformation $x=x'+1$, $y=y'$, and then apply the theorem to the resulting equation.

Also solved by HARRY LANGMAN, E. C. KIEFER, A. PELLETIER, PAUL WERNICKE, AND THE PROPOSER.

3284 [1927, 438]. Proposed by Burrell Morgan, Krellitz, W. Va., and Norman Anning, University of Michigan.

In a given sphere is inscribed the maximum right prism whose bases are regular polygons of a given number of sides. Show that its altitude is independent of the number of its lateral faces.

SOLUTION BY A. PELLETIER, Montreal, Canada.

Let r be the radius of the sphere, n the number of sides in the polygon, and x the radius of the circle circumscribed about the polygon. Then the volume of the prism is expressed by

$$nx^2 (r^2 - x^2)^{1/2} \sin \left(\frac{2\pi}{n}\right).$$

The maximum of this expression takes place when $x^2=2r^2/3$.

Hence the altitude $a=2(r^2-x^2)^{1/2}=2\cdot 3^{-1/2}r$, a value independent of n, the number of lateral faces of the prism.

Also solved by H. C. BRADLEY, HARRY LANGMAN, E. C. KIEFER, W. T. REID, AND J. B. REYNOLDS.

NOTES AND NEWS

LINNAEUS WAYLAND DOWLING, Professor of Mathematics at the University of Wisconsin, died at his home in Madison on September 16. He had been failing in health for some months, and was obliged to relinquish his work early in the second semester of the past year.

Professor Dowling was born at Medina, Michigan, on December 8, 1867. He was a graduate of Adrian College, and in 1895 received the degree of Doctor of Philosophy from Clark University, his doctor's dissertation being "On the forms of plane quintic curves." In the same year he came to the University of Wisconsin as instructor in mathematics, remaining a member of the department until his last illness. In 1905–6 he spent a year at Turin studying under Segré. He was a member of the American Mathematical Society, the Mathematical Association of America, and the Circolo Matematico di Palermo. He was the author of textbooks on analytic geometry, projective geometry, and mathematics of insurance, and edited the mathematical series for the Johnson Publishing Company.

This long service as teacher and friend has been rich in its fruitful associations. Magnetic in personality, genial of manner, peculiarly endowed with the teaching gift, he quickly won and maintained the admiration and devotion of his students. He brought to them a rare combination of discipline and vision, a mingling of carefully developed fundamentals with glimpses of the heights beyond. Many of his students who later became teachers testify to the special inspiration and help in their work received in his classroom.

The days of growing weakness were brightened by daily visits of friends and students, who found him, as always, full of sympathy for their interests. His home was a shrine, to which flocked the many who loved him. These found him with peace and serenity of spirit, surrounded by the things he put first, his family, his friends, and his books. Active in community life, lover of literature, music, and nature, his passing leaves a wide sense of personal loss.

The annual meeting of the National Council of Teachers of Mathematics will be held at the Statler Hotel in Cleveland, Ohio, Feb. 22 and 23, 1929. Mr. Harry C. Barber of The Phillips Exeter Academy is the president of the organization and he will be pleased to answer any inquiries concerning the annual meeting or any other phase of the Council's activity. It may be of interest to note that the National Council has just become an incorporate body and that there is great activity on foot to increase the membership from the present five thousand to at least ten thousand in the next two years. Its official journal is *The Mathematical Teacher*, which is the only journal in this country devoted exclusively to mathematics in the elementary and secondary fields; and it is

highly worth while for every teacher of secondary mathematics to be a member of the Council and a reader of this journal.

At its commencement exercises in June, Dartmouth College conferred the honorary degree of Doctor of Science on Professor CHARLES N. HASKINS of the department of mathematics. This was an expression of appreciation for his work in connection with the building of the new Baker Memorial Library dedicated at that time.

Assistant Professor FRED W. BEAL, of the University of Pennsylvania, has been promoted to a professorship of mathematics.

Assistant Professor C. C. BRAMBLE has been promoted to a professorship of mathematics in the post graduate school of the United States Naval Academy.

Associate Professor J. A. BULLARD, of the United States Naval Academy, has been appointed to a professorship of mathematics at the University of Vermont.

Dr. A. H. COPELAND has been appointed assistant professor of mathematics at the University of Buffalo.

Professor J. C. FITTERER has been appointed associate professor of mathematics at the Colorado School of Mines.

Dr. H. C. HICKS has been appointed assistant professor of mathematics at the University of Oregon.

Assistant Professor DANIEL HULL, of the University of Notre Dame, has been promoted to the rank of professor and head of the department of physics.

Assistant Professor ROGER A. JOHNSON, of Hunter College of the City of New York, has been promoted to an associate professorship of mathematics.

Assistant Professor JOHN R. KLINE, of the University of Pennsylvania, has been promoted to a professorship of mathematics.

Dr. E. E. LIBMAN, of the University of Illinois, has been promoted to as assistant professorship of physics.

Associate Professor J. J. NASSAU has resumed his work at the Case School of Applied Science after a year's leave of absence spent at Cambridge University working principally under Professor Eddington.

The Reverend E. C. PHILLIPS, director of the astronomical observatory of Georgetown University, has been appointed to the office of provincial of the Jesuit Province of Maryland-New York.

Dr. WILLIAM C. RISSELMAN, of the University of Minnesota, has been appointed assistant professor of mathematics in the University of Pittsburg. His work will be at the Junior College branch of the University at Uniontown, Pa.

Associate Professor H. M. ROBERTS, of the United States Naval Academy, has been promoted to a professorship of mathematics.

Assistant Professor J. B. ROSENBACH, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

EDWIN W. SCHREIBER, formerly head of the mathematics department, Proviso Township High School, Maywood, Illinois, is continuing his graduate studies under Professor Karpinski at the University of Michigan. Mr. Schreiber has been honored in being appointed to a university fellowship for the year 1928–1929.

At the University of California at Los Angeles, Assistant Professor H. M. SHOWMAN has been made recorder of the university but will remain a member of the department of mathematics.

Assistant Professor STANLEY P. SHUGERT, of the University of Pennsylvania, has been promoted to a professorship of mathematics.

Professor EVAN THOMAS, of the University of Vermont, has retired.

The following appointments to instructorships in mathematics are announced:

Albion College, Mr. WATSON M. DAVIS;
University of Arkansas, Mr. HARVEY A. WRIGHT;
Brown University, Mr. H. S. THURSTON;
Milton College, Miss FANNIE HOPKINS;
University of Pennsylvania, V. W. ADKISSON, P. A. KNEDLER;
St. Olaf's College, Mr. CLARENCE CARLSON.

CORRIGENDA

The following corrections should be made in the August-September issue of the Monthly:

On page 335, in line 8, the statement, "The words were written by Miss Lutz" should be replaced by this sentence: "The pageant was written by Miss Banes, assisted by five students of her class in the history of mathematics."

On page 361, in line 19, change 7.973 to 7.972; and in the fourth line from the bottom, change -5.011 to -5.010.

HART'S

MATHEMATICS OF INVESTMENT

By William L. Hart, Ph.D., Professor of Mathematics in the University of Minnesota

A masterly treatment of the principles of investment for college students of business administration or practical mathematics. It provides a course in the theory and applications of annuities certain and in the mathematical aspect of life insurance. The book is adaptable to a longer or a shorter course.

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Adopted by the University of Michigan, University of Minnesota, Mt. Holyoke College, University of Cincinnati, and many others.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

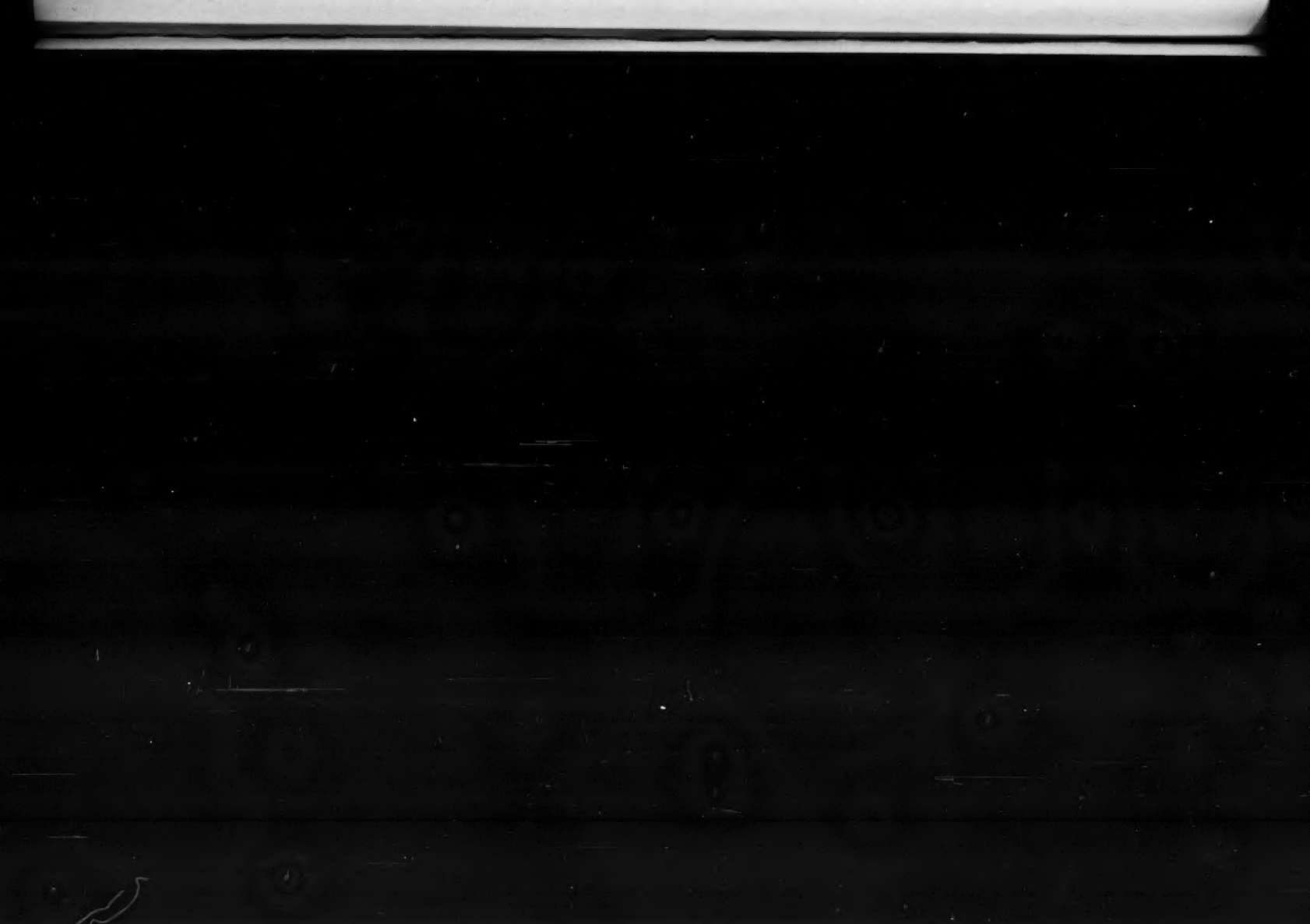
MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December 28-29, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.
INDIANA, Butler University, May 11-12.
IOWA, Grinnell College, May 4-5.
KANSAS, Topeka, Kan., February 4.
KENTUCKY, Louisville, Ky., April 21.
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.
MICHIGAN, Ann Arbor, Mich., March 31.
MINNESOTA, St. Joseph, Minn., May 19.
MISSOURI, Kansas City, Mo., November.
NEBRASKA, Midland College, April 28.
OHIO, Columbus, Ohio, April 5.
PHILADELPHIA, Philadelphia, Pa., December 1.
ROCKY MOUNTAIN, Golden, Colo., April 20-21.
SOUTHEASTERN, Dunham, N. C., March.
SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.
TEXAS, Texas A. and M. College, Jan. 28.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.



THE THIRTEENTH SUMMER MEETING OF THE ASSOCIATION.

The thirteenth summer meeting of the Mathematical Association of America was held, by invitation, at Amherst College on Monday and Tuesday, September 3–4, 1928, in conjunction with the summer meeting and the colloquium of the American Mathematical Society. 183 persons were present, including 111 members of the Association:

C. R. ADAMS, Brown University.
ELI ALLISON, Brenau College.
ETHEL L. ANDERTON, Smith College.
H. E. ARNOLD, Wesleyan University.

CLARA L. BACON, Goucher College.
A. A. BENNETT, Brown University
H. F. BLICHFELDT, Stanford University.
THEODORE BENNETT, University of Illinois.
G. A. BLISS, University of Chicago.
HENRY BLUMBERG, Ohio State University.
J. W. BRADSHAW, University of Michigan.
H. S. BROWN, Hamilton College.

W. D. CAIRNS, Oberlin College.
B. H. CAMP, Wesleyan University.
C. C. CAMP, University of Nebraska.
G. A. CAMPBELL, Amer. Tel. & Tel. Co., New York, N. Y.
MILDRED E. CARLEN, Brown University.
F. E. CARR, Oberlin College.
A. B. COBLE, University of Illinois.
ABRAHAM COHEN, Johns Hopkins University.
J. L. COOLIDGE, Harvard University.
LENNIE A. COPELAND, Wellesley College.
A. R. CRATHORNE, University of Illinois.
C. H. CURRIER, Brown University.

MARGUERITE DARKOW, Philadelphia, Pa.
H. T. DAVIS, Indiana University.
F. F. DECKER, Syracuse University.
L. S. DEDERICK, Aberdeen Proving Ground.
L. L. DINES, University of Saskatchewan.
H. A. DOBELL, N. Y. State College for Teachers.
ELEANOR C. DOAK, Mount Holyoke College.
H. L. DORWART, Williams College.
ARNOLD DRESDEN, Swarthmore College.

T. C. ESTY, Amherst College.

W. B. FORD, University of Michigan.
TOMLINSON FORT, Lehigh University.
T. C. FRY, Bell Telephone Laboratories, New York, N. Y.

H. M. GEHMAN, Yale University.
D. C. GILLESPIE, Cornell University.
J. W. GLOVER, University of Michigan.
V. G. GROVE, Michigan State College.

MAY N. HARWOOD. Syracuse University.
MARY G. HASEMAN, Junior College of Connecticut.
E. R. HEDRICK, University of California at Los Angeles.
A. O. HICKSON, Brown University.
C. M. HUBER, Rutgers College.
W. A. HURWITZ, Cornell University.

LOUIS INGOLD, University of Missouri.
M. H. INGRAHAM, University of Wisconsin.

DUNHAM JACKSON, University of Minnesota.
R. L. JEFFERY, Acadia University.
MYRA I. JOHNSON, New York, N. Y.
R. A. JOHNSON, Hunter College.
F. E. JOHNSTON, University of Illinois.

HERMAN KARNOW, University of Colorado.
A. J. KEMPNER, University of Colorado.
LYLAH KRYDER, Hunter College.

R. E. LANGER, University of Wisconsin.
HARRY LANGMAN, New York, N. Y.
C. G. LATIMER, University of Kentucky.
D. D. LEIB, Connecticut College for Women.

C. C. MACDUFFEE, Ohio State University.
W. D. MACMILLAN, University of Chicago.
ISRAEL MAIZLISH, Centenary College.
MRS. YETTA V. MAIZLISH, Shreveport, La.
H. P. MANNING, Brown University.
EMILIE N. MARTIN, Mount Holyoke College.
J. N. MICHIE, Texas Technical College.
W. E. MILNE, University of Oregon.
E. B. MODE, Boston University.
C. L. E. MOORE, Massachusetts Institute of Technology.
F. C. MOORE, Massachusetts Agricultural College.